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ORIGINAL ARTICLE

Ivan Argatov

The modified Kachanov method. Evaporation of multiple droplets

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Abstract Evaporation of multiple sessile droplets deposited on an impermeable flat substrate is considered in the diffusion-limited isothermal regime, with particular emphasis on determining the total quasi-stationary vapor fluxes from the droplet surfaces. The proposed approximate solution is based on Kachanov's approximation for the vapor concentration field in the semi-infinite air domain, expressed as a linear combination of solutions to single-droplet problems, with the coefficients determined by imposing appropriate Bubnov–Galerkin orthogonality relations to enforce the boundary conditions on the droplet surfaces. A comparison of the proposed modified Kachanov method with other approximate approaches is presented. The case of multiple thin circular sessile droplets is examined in detail.

Keywords Kachanov's method · Multiple evaporation · Sessile droplets · Contact interaction · Asymptotic modeling

1 Introduction

The interaction between multiple evaporating sessile droplets on a flat surface is of significant practical interest, and recent advances in both experimental and theoretical studies have been reviewed in [1]. Important analytical approximations for the competitive diffusion-limited evaporation of multiple sessile droplets were derived by Wray *et al.* [2] in the thin-droplet limit and by Masoud *et al.* [3] for spherical-cap droplets. Recent work [2,3] has clarified how neighboring droplets compete for vapor, leading to non-uniform evaporative fluxes. For thin sessile droplets, this interaction produces spatially varying "shielding," reducing evaporation in regions where droplets are closest.

On both the single- and multi-droplet levels, much analytical and semi-analytical work continues to rely on Maxwell's classical quasi-steady solution for diffusion-limited mass transfer from a spherical droplet — adopted here as well — extended to incorporate interfacial effects [4]. Comprehensive reviews [5,6] show how such solutions are embedded in larger modelling frameworks, including discrete and quasi-discrete approaches, with extensions to transient and multidimensional internal transport.

As it was shown earlier [7,8], the problem of evaporation in the diffusion-limited regime reduces to a harmonic potential problem: the solution of the Laplace equation in a parametrically time-dependent semi-infinite domain occupied by air with the Dirichlet boundary condition on the droplet surface, a mathematical problem formulation that (independently of its physical context) has been widely studied in the literature

I. Argatov
Department of Biomedical Science, Malmö University, Jan Waldenströms gata 25, Malmö SE-205 06, Sweden

I. Argatov (✉)
Biofilms – Research Center for Biointerfaces, Malmö University, Per Albin Hanssons väg 35, Malmö SE-205 06, Sweden
E-mail: ivan.argatov@mau.se

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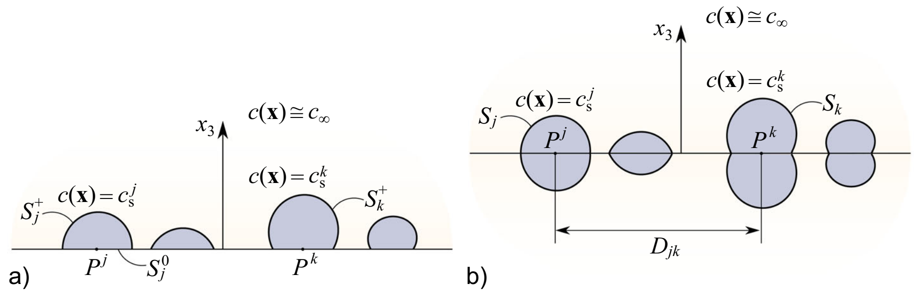

Fig. 1 a) Schematic of multiple droplets (side view) in a semi-infinite air domain; b) An infinite "air domain" obtained by applying a symmetry reflection

for canonical domains. In particular, for a single spherical-cap sessile droplet, an exact analytical solution is available in different forms [9,10].

Analytical treatments of droplet–droplet interaction typically assume diffusion-limited evaporation in quiescent air and convective effects are negligible. Under these assumptions, superposition and multipole methods from potential theory apply, with droplets represented as Dirichlet boundary patches, yielding interaction coefficients or shielding factors as functions of droplet size and spacing [11]. In simple geometries, the resulting interaction-corrected evaporation rates can be expressed via asymptotic expansions for well-separated or nearly touching droplets [12].

It goes without saying that a general strategy for solving the problem of multiple-droplet evaporation is to reduce it to a sequence of single-droplet problems. However, it can be shown (see, e.g., [13]) that the straightforward application of an iterative method is accompanied by an error that grows with the number of droplets in the system, and the multiple-droplet problem prompts the development of special approaches that account for interaction and cooperative effects.

Foldy [14] appears to have been the first to consider multiple scattering of scalar (e.g., acoustic) waves [15] and to introduce a self-consistent method that results in solving a system of simultaneous linear algebraic equations for the scattering amplitudes. The problem of multiple-droplet evaporation is, in this sense, analogous to multiple scattering by a cluster of sound-soft obstacles, particularly in the long-wavelength limit [16], and asymptotic modeling approaches developed in that context [17,18] can be directly applied to its analysis. To effectively approximate potentials for bodies containing clusters of small defects, the method of mesoscale asymptotic approximations was developed [19,20].

Holm [21] was probably the first to observe a strong interaction effect between contact spots on the constriction resistance of a cluster of electrical microcontacts, which markedly differs from the non-interaction approximation. This microcontact interaction effect was later studied in a number of publications [22–24] following the seminal paper by Greenwood [25]. The problem of multiple evaporation of thin sessile droplets is, to a certain degree, equivalent to the problem of multiple frictionless contacts or to the electrostatics problem for a system of infinitesimally thin disks located in a single plane. This analogy motivates the application of the Kachanov method [26] to its analysis, though this method has been developed for multiple contacts.

Kachanov's method was originally introduced in the context of crack interactions [27,28] and is based on self-consistency relations linking the average tractions acting on individual cracks. The method was later adapted for the approximate analysis of frictionless contact problems involving multiple punches indenting an elastic half-space [26]. More recently, it has been extended to multi-punch contact interface problems [29] and to multiple contact problems involving bonded punches [30].

Here, we develop a modified Kachanov method applied to the problem of multiple evaporation.

2 Problem of multiple evaporation

We consider N liquid droplets on the flat surface, $x_3 = 0$, of an impermeable substrate, which are exposed to a quiescent air, $x_3 > 0$ (outside the droplets), referred to a Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$. Let S_j^0 and S_j^+ denote respectively the contact area and the free surface area of the j-th droplet (see Fig. 1a).

We adopt the model of diffusion-limited quasi-steady evaporation, assuming that the diffusion of the vapor from a near-surface layer is slower than the evaporation itself [7,31]. In addition, the evaporation process is assumed to be isothermal (the temperature of the air is equal to that of the substrate), thereby also neglecting the evaporative cooling effect [32]. Yet one more important aspect of modeling the evaporation process is the assumption about the evaporation modes that concern pinning or depinning of the droplet edges [1], which, however, is not involved in the subsequent analysis of the static picture of multiple evaporative interaction between the droplets.

Let $c(\mathbf{x})$ denote the vapor concentration in the air, such that

$$c(\mathbf{x}) \to c_\infty \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{1}$$

where c_∞ is the vapor concentration far away from the array of droplets, $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, and the infinite-size idealization is assumed for the air domain.

At the surface of the j-th droplet, the vapor concentration is equal to the saturation value, c_{s}^j, so that

$$c(\mathbf{x}) = c_{\mathrm{s}}^j, \quad \mathbf{x} \in S_j^+ \quad (j = 1, 2, \ldots, N). \tag{2}$$

We take a broad view here by allowing the constants $c_{\mathrm{s}}^1, c_{\mathrm{s}}^2, \ldots, c_{\mathrm{s}}^N$ to be different not for the sake of generality but rather to facilitate comparison with other methods.

In the air domain, the function $c(\mathbf{x})$ solves the Laplace equation $\nabla_x^2 c(\mathbf{x}) = 0$, where $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ is the gradient operator, whereas on the exposed surface of the substrate (i.e., on the surface free from droplets), it satisfies the boundary condition

$$\frac{\partial c}{\partial x_3}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \cup_{j=1}^N \bar{S}_j^0. \tag{3}$$

According to Fick's first law of diffusion, the diffusion flux is defined as $\mathbf{j}(\mathbf{x}) = -D\nabla_x c(\mathbf{x})$, where D is the diffusion coefficient of vapor in the air. Let also $\mathbf{n}^j(\mathbf{x})$ denote the unit normal vector to the surface S_j^+, which is directed into the air domain. Since the vapor concentration $c(\mathbf{x})$ is uniform over the entire free surface S_j^+ of the j-th droplet (see Eq. (2)), then the vector $\mathbf{j}(\mathbf{x})$ at any point $\mathbf{x} \in S_j^+$ is parallel to $\mathbf{n}^j(\mathbf{x})$, so that $\mathbf{j}(\mathbf{x}) = Dq_j(\mathbf{x})\mathbf{n}^j(\mathbf{x})$, where we have introduced the notation

$$q_j(\mathbf{x}) = -\mathbf{n}^j(\mathbf{x}) \cdot \nabla_x c(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in S_j^+, \tag{4}$$

and the dot denotes the scalar product.

The integral of the normal vapor flux $\mathbf{n}^j(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x})$ over the droplet free surface determines the total evaporative mass flow rate from the j-th droplet, $J_j = DQ_j$, where

$$Q_j = -\iint_{S_j^+} \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x c(\mathbf{x})\, \mathrm{d}\sigma_x \tag{5}$$

and $\mathrm{d}\sigma_x$ is the surface area element.

The problem of multiple droplet evaporation is formulated as follows: Find a harmonic potential $c(\mathbf{x})$ that satisfies the boundary conditions (2) and (3), as well as the asymptotic condition (1). The quantities $q_j(\mathbf{x})$ and Q_j, defined by (4) and (5), are of primary importance.

3 The modified Kachanov method

We take advantage of the homogeneous Neumann boundary conditions (3) and consider the equivalent problem obtained by reflecting the geometrical air domain across the substrate plane $x_3 = 0$ (see Fig. 1b) and replacing the Dirichlet boundary conditions (2) with the following extended one:

$$c(\mathbf{x}) = c_{\mathrm{s}}^j, \quad \mathbf{x} \in S_j \quad (j = 1, 2, \ldots, N). \tag{6}$$

Here, S_j is the surface with the mirror symmetry such that S_j coincides with S_j^+ in the upper half-space $x_3 > 0$.

Let us now introduce the auxiliary normalized harmonic potential $\phi_j(\mathbf{x})$ that solves the j-th single-droplet evaporation problem and satisfies the following boundary and asymptotic conditions:

$$\phi_j(\mathbf{x}) = 1, \quad \mathbf{x} \in S_j; \qquad \phi_j(\mathbf{x}) = o(1), \quad |\mathbf{x}| \to \infty. \tag{7}$$

Indeed, the quasi-static evaporation of the j-th droplet in the absence of the other droplets is described by the vapor concentration $c(\mathbf{x}) = c_\infty + (c_s^j - c_\infty)\phi_j(\mathbf{x})$.

In the case of multiple droplets, we make use of Kachanov's approximation

$$c(\mathbf{x}) = c_\infty + \sum_{k=1}^{N} d_k \phi_k(\mathbf{x}) \tag{8}$$

with unknown constant coefficients $d_1, d_2, \ldots, d_N$ that should be determined via approximately satisfying the boundary conditions (3), since, in view of (7), the asymptotic condition (1) has already in place.

So, the j-th boundary condition (6) will be satisfied in a certain average sense, if the following condition has been enforced:

$$\iint_{S_j} \Big(c_s^j - c_\infty - d_j - \sum_{k \neq j} d_k \phi_k(\mathbf{x}) \Big) \frac{\partial \phi_j}{\partial n}(\mathbf{x})\, \mathrm{d}\sigma_x = 0. \tag{9}$$

Here, $\partial\phi_j(\mathbf{x})/\partial n = \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x \phi_j(\mathbf{x})$ is the normal derivative of the potential $\phi_j(\mathbf{x})$.

We recall that the harmonic function that meets the normalized Dirichlet boundary condition (see (7)) is called the conductor potential [33] and possesses the properties

$$\phi_j(\mathbf{x}) = \frac{\mathsf{c}_j}{|\mathbf{x}|} + o(|\mathbf{x}|^{-1}), \quad |\mathbf{x}| \to \infty, \tag{10}$$

$$\mathsf{c}_j = -\frac{1}{4\pi} \iint_{S_j} \frac{\partial \phi_j}{\partial n}(\mathbf{x})\, \mathrm{d}\sigma_x, \tag{11}$$

where c_j is the harmonic (or electrostatic) capacity of S_j.

It should be noted that according to the maximum principle for harmonic functions (see, e.g., [34]), the normal derivative of $\phi_j(\mathbf{x})$, which appears under the integrals in (9) and (11), is strictly negative.

In view of (11), Eq. (9) can be represented in the form

$$d_j + \sum_{k \neq j} \tilde{\Lambda}_{kj} d_k = c_s^j - c_\infty, \tag{12}$$

where we have introduced the notation

$$\tilde{\Lambda}_{kj} = -\frac{1}{4\pi \mathsf{c}_j} \iint_{S_j} \phi_k(\mathbf{x}) \frac{\partial \phi_j}{\partial n}(\mathbf{x})\, \mathrm{d}\sigma_x. \tag{13}$$

The dimensionless positive coefficients $\tilde{\Lambda}_{kj}$, defined by (13), will be called the modified transmission factors (responsible for the influence of the k-th droplets on the j-th droplet).

In order to evaluate the rate coefficient Q_j for the total evaporative flux from the j-th droplet, we substitute the Kachanov approximation (8) into Eq. (5) and arrive at the approximate equation

$$2Q_j = -d_j \iint_{S_j} \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x \phi_j(\mathbf{x})\, \mathrm{d}\sigma_x - \sum_{k \neq j} d_k \iint_{S_j} \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x \phi_k(\mathbf{x})\, \mathrm{d}\sigma_x, \tag{14}$$

from where, in view of (11), it follows that

$$Q_j = 2\pi \mathsf{c}_j d_j. \tag{15}$$

We note that the last integrals on the right-hand side of Eq. (14) vanishes because the functions $\phi_k(\mathbf{x})$, $k \neq j$, are harmonic inside the surface S_j.

Finally, considering (15), we can rewrite Eq. (12) as

$$Q_j + \sum_{k \neq j} \frac{\mathsf{c}_j}{\mathsf{c}_k} \tilde{\Lambda}_{kj} Q_k = 2\pi \mathsf{c}_j (c_s^j - c_\infty), \tag{16}$$

thereby getting rid of the auxiliary unknowns $d_1, d_2, \ldots, d_N$.

Remark 1 By the maximum principle and the Hopf boundary point lemma for harmonic functions (see, e.g., [34]), since $\phi_j(\mathbf{x}) = 1$ on S_j, it follows that $\phi_j(\mathbf{x}) > 0$ for all $\mathbf{x}$ in the exterior of S_j, and that the outward normal derivative satisfies the inequality $\partial\phi_j/\partial n(\mathbf{x}) < 0$ on S_j. In the multiple-droplet evaporation problem, we assume identical saturation conditions on all droplet surfaces, namely $c_{\rm s}^j = c_{\rm s} = {\rm const}$, with $c_{\rm s} > c_\infty > 0$. The vapor concentration field can then be written in the form $c(\mathbf{x}) = (c_{\rm s} - c_\infty)\phi(\mathbf{x})$, where $\phi(\mathbf{x}) = 1$ on $S_1 \cup \ldots \cup S_N$ and $\phi(\mathbf{x}) \to 0$ at infinity. Under these assumptions, the evaporative flux from each droplet is strictly positive, and hence $Q_j > 0$ $(j = 1, 2, \ldots, N)$. Using the approximate relation (15), this implies that the coefficients d_k appearing in formula (8) must be positive. A rigorous *a priori* proof of the positivity of these coefficients will be addressed elsewhere.

Remark 2 Let us introduce the notation

$$\tilde{\lambda}_{kj} = -\frac{1}{4\pi}\iint\limits_{S_j} \phi_k(\mathbf{x})\frac{\partial\phi_j}{\partial n}(\mathbf{x})\,{\rm d}\sigma_x. \tag{17}$$

By applying Green's first identity, the surface integral on the right-hand side of (17) can be transformed into the following volume integral over the domain Ω_{jk}, defined as the region exterior to the surfaces S_j and S_k:

$$\tilde{\lambda}_{kj} = \frac{1}{4\pi}\iiint\limits_{\Omega_{jk}} \nabla_x\phi_k(\mathbf{x})\cdot\nabla_x\phi_j(\mathbf{x})\,{\rm d}\mathbf{x}. \tag{18}$$

Consequently, in view of (11) and (17), Eq. (12) can be rewritten in the matrix form

$$\Lambda\mathbf{d} = \mathbf{c}, \tag{19}$$

where $\mathbf{d} = (d_1, d_2, \ldots, d_N)^{\mathsf{T}}$, $\mathbf{c} = (\mathsf{c}_1(c_{\rm s}^1 - c_\infty), \mathsf{c}_2(c_{\rm s}^2 - c_\infty), \ldots, \mathsf{c}_N(c_{\rm s}^N - c_\infty))^{\mathsf{T}}$, T is the transposition symbol, and Λ is a symmetric matrix with entries $\tilde{\lambda}_{kj}$ given by (18). This formulation allows one to establish the solvability of the resulting linear algebraic system (16).

Finally, the surface integral in (17) may be interpreted as a duality pairing between the functional spaces $H^{-1/2}(S_j)$ and $H^{1/2}(S_j)$. As noted by the anonymous reviewer, the derivation of Eq. (9) via a generalized weighted-residual formulation therefore corresponds more closely to what is commonly referred to in the literature as a Petrov–Galerkin method (see [35], Section 7.2). Here, in referring to the Bubnov–Galerkin method, we merely stress the underlying principle of enforcing the governing equation through appropriate orthogonality relations.

4 Comparison with other approximate methods

In this section, without claiming to be exhaustive, we briefly consider other approximate solution approaches in application to the multiple evaporation problem.

4.1 Asymptotic modeling

In the case of relatively small (widely separated) droplets, we can apply one or another asymptotic method. With this aim, a small positive parameter, usually denoted by ε, is introduced into the problem, e.g., as the ratio of the maximum droplet diameter to the minimum inter-droplet distance (as measured between the centers of droplets). To be specific, let P^j denote the center of the j-th droplet, which is located on the substrate plane. To be more precise, we assume that P^j coincides with the center of mass of the surface S_j with the mass density $-\partial\phi_j(\mathbf{x})/\partial n$.

Here we employ our favorite asymptotic modeling approach based on the method of matched asymptotic expansions [36,37] and construct two types of asymptotic approximations for $c(\mathbf{x})$, namely, one of them is the so-called outer asymptotic expansion

$$c^{\rm out}(\mathbf{x}) = c_\infty + \sum_{k=1}^{N} A_k\Phi(\mathbf{x} - P^k), \tag{20}$$

where $A_1, A_2, \dots, A_N$ are some constant coefficients, and

$$\Phi(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|} \tag{21}$$

is the fundamental solution of the Laplace equation corresponding to a point source located at $\mathbf{y}$. Due to the presence of singularities in (20), the outer asymptotic approximation is supposed to be valid far away from the droplets (and is also called "far field").

Near the j-th droplet, we construct the inner asymptotic approximation, $c^{\rm in}(\mathbf{x})$, also sometimes called "boundary layer". Strictly speaking, the boundary-layer asymptotic technique relies on the introduction and use of the stretched coordinates $\boldsymbol{\xi}^j = \varepsilon^{-1}(\mathbf{x} - P^j)$, a step that will be skipped in the present analysis, and we put

$$c_j^{\rm in}(\mathbf{x}) = (c_{\rm s}^j - c_{\rm a}^j)\phi_j(\mathbf{x}) + c_{\rm a}^j, \tag{22}$$

where $c_{\rm a}^j$ is some unknown (asymptotic matching) constant, $j = 1, 2, \dots, N$.

The substitution of (22) into Eq. (5) yields the relation

$$Q_j = 2\pi \mathsf{c}_j (c_{\rm s}^j - c_{\rm a}^j). \tag{23}$$

By virtue of (10) and the special assumption on the location of the center P^j of the surface S_j (see also [38]), we have

$$\phi_j(\mathbf{x}) = 4\pi \mathsf{c}_j \Phi(\mathbf{x} - P^j) + O(|\mathbf{x} - P^j|^{-3}), \quad |\mathbf{x} - P^j| \to \infty, \tag{24}$$

where the first term is of the order $O(|\mathbf{x} - P^j|^{-1})$.

As the result of asymptotic matching of the asymptotic expansions (20) and (22), in view of (23), we arrive at the relations

$$A_j = 4\pi \mathsf{c}_j (c_{\rm s}^j - c_{\rm a}^j), \tag{25}$$

$$c_{\rm a}^j = c_\infty + \sum_{k \neq j} A_k \Phi(P^j - P^k), \tag{26}$$

from where, taking (23) into account, we derive the system of linear algebraic equations ($j = 1, 2, \dots, N$)

$$Q_j + \sum_{k\neq j} \grave{\Lambda}_{kj} Q_k = 2\pi \mathsf{c}_j (c_{\rm s}^j - c_\infty), \tag{27}$$

where we have introduced the notation

$$\grave{\Lambda}_{kj} = \frac{\mathsf{c}_k}{|P^j - P^k|}. \tag{28}$$

By comparing Eqs. (16) and (27), in view of (28), we arrive the the approximate relation

$$\tilde{\Lambda}_{kj} \simeq \frac{\mathsf{c}_k}{|P^j - P^k|}, \tag{29}$$

which, in light of (11), (13), and (24), can be shown to be asymptotically exact.

Remark 3 Let us introduce the unknown densities

$$\mu_j(\mathbf{x}) = \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x c(\mathbf{x}) \equiv \frac{\partial c}{\partial n}(\mathbf{x}), \quad \mathbf{x} \in S_j. \tag{30}$$

Then, by a classical approach, using the fundamental solution $\Phi(\mathbf{x} - \mathbf{y})$ and Green's second identity, the multiple-obstacle problem can be reduced to the system of boundary integral equations

$$\sum_{k=1}^{N} \iint_{S_k} \Phi(\mathbf{x}-\mathbf{y})\mu_k(\mathbf{y})\, d\sigma_y = -\frac{1}{2}(c_{\rm s}^j - c_\infty) + \sum_{k=1}^{N} (c_{\rm s}^k - c_\infty) \iint_{S_k} \frac{\partial \Phi}{\partial n_y}(\mathbf{x}-\mathbf{y})\, d\sigma_y, \tag{31}$$

where $\mathbf{x} \in S_j$ $(j = 1, 2, \ldots, N)$.

By fixing the index j, we equivalently rewrite Eq. (31) in the form

$$\iint_{S_j} \Phi(\mathbf{x} - \mathbf{y})\mu_j(\mathbf{y})\,\mathrm{d}\sigma_y = -(c_\mathrm{s}^j - c_\infty) - \sum_{k \neq j} \iint_{S_k} \Phi(\mathbf{x} - \mathbf{y})\mu_k(\mathbf{y})\,\mathrm{d}\sigma_y, \tag{32}$$

where the Gaussian formula for the double-layer potential has been employed.

Now, by taking $\mathbf{x} = P^j$, the integrals in the last (interaction) term on the right-hand side of Eq. (32), in view of (5) and (30), can be approximated as

$$\iint_{S_k} \Phi(\mathbf{x} - \mathbf{y})\mu_k(\mathbf{y})\,\mathrm{d}\sigma_y \simeq \iint_{S_k} \Phi(P^j - \mathbf{y})\mu_k(\mathbf{y})\,\mathrm{d}\sigma_y$$
$$\simeq \Phi(P^j - P^k) \iint_{S_k} \mu_k(\mathbf{y})\,\mathrm{d}\sigma_y = -2Q_k\Phi(P^j - P^k). \tag{33}$$

So, by making use of the leading-order asymptotic expansion (33), the multiple-obstacle problem (32) can be reduced to a system of single-obstacle problems

$$\iint_{S_j} \Phi(\mathbf{x} - \mathbf{y})\mu_j(\mathbf{y})\,\mathrm{d}\sigma_y \simeq -(c_\mathrm{s}^j - c_\infty) + 2\sum_{k \neq j} Q_k\Phi(P^j - P^k) \quad (j = 1, 2, \ldots, N) \tag{34}$$

with constant right-hand sides, and therefore, we readily obtain that

$$\mu_j(\mathbf{x}) \simeq \left((c_\mathrm{s}^j - c_\infty) - 2\sum_{k \neq j} Q_k\Phi(P^j - P^k)\right)\frac{\partial \phi_j}{\partial n}(\mathbf{x}), \quad \mathbf{x} \in S_j. \tag{35}$$

Finally, in view of (15) and (35), we arrive at the relation

$$Q_j \simeq 2\pi \mathsf{c}_j\left((c_\mathrm{s}^j - c_\infty) - 2\sum_{k \neq j} Q_k\Phi(P^j - P^k)\right). \tag{36}$$

Up to notation (see (21) and (28)), formula (36) agrees exactly with Eq. (27).

A somewhat similar line of reasoning was used by Greenwood [25] in defining his approximate solution to the multiple-contact problem.

4.2 Source-function-based approximation

For an isolated surface S_j, we introduce the source function, $G_j(\mathbf{x}, \mathbf{y})$, (also called Green's function) of the following external Dirichlet problem for the Laplace equation considered outside of S_j:

$$-\nabla_x^2 G_j(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}); \qquad G_j(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in S_j; \tag{37}$$

$$G_j(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) + g_j(\mathbf{x}, \mathbf{y}). \tag{38}$$

Here, $\Phi(\mathbf{x} - \mathbf{y}) = (4\pi)^{-1}|\mathbf{x} - \mathbf{y}|^{-1}$ and $g_j(\mathbf{x}, \mathbf{y})$ are the singular and regular parts of the source function.

In a region near the surface S_j (which is small enough not to include any other surface S_k for $k \neq j$), departing from the single-droplet solution, we approximate the solution of the multiple evaporation problem as

$$c_j^\mathrm{loc}(\mathbf{x}) = (c_\mathrm{s}^j - c_\infty)\phi_j(\mathbf{x}) + c_\infty + \sum_{k \neq j} A_k G_j(\mathbf{x}, P^k), \tag{39}$$

where $A_1, A_2, \ldots, A_N$ are some constant coefficients.

By regarding all other droplets except the j-th one as point sources, we will have

$$Q_j = \frac{A_j}{2}. \tag{40}$$

On the other hand, the substitution of (39) into Eq. (5), in view of (40), yields

$$Q_j + \sum_{k \neq j} \lambda_{kj} Q_k = 2\pi \mathsf{c}_j (c_s^j - c_\infty), \tag{41}$$

where we have introduced the notation

$$\lambda_{kj} = \iint\limits_{S_j} \frac{\partial G_j}{\partial n}(\mathbf{x}, P^k)\, d\sigma_x. \tag{42}$$

It is important to note here that, by the application of Green's second identity for the combination of functions $G_j(\mathbf{x}, P^k)$ and $\phi_j(\mathbf{x})$, formula (42) can be equivalently simplified as

$$\lambda_{kj} = \phi_j(P^k). \tag{43}$$

Further, by comparing Eqs, (16) and (41), in view of (29) and (43), we arrive at the approximate relations

$$\frac{\mathsf{c}_j}{\mathsf{c}_k}\tilde{\Lambda}_{kj} \simeq \lambda_{kj}, \quad \phi_j(P^k) \simeq \frac{\mathsf{c}_j}{|P^j - P^k|}, \tag{44}$$

where the second one immediately follows from (24).

The source-function-based approximation was first introduced by Galin [39] in the framework of multiple contact problems.

4.3 Reciprocal-relation-based approximation

By applying Green's second identity for the functions $c(\mathbf{x}) - c_\infty$ and $\phi_j(\mathbf{x})$, we readily get

$$\iint\limits_{S_j} \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x c(\mathbf{x})\, d\sigma_x + \sum_{k \neq j} \iint\limits_{S_k} \phi_j(\mathbf{x}) \mathbf{n}^k(\mathbf{x}) \cdot \nabla_x c(\mathbf{x})\, d\sigma_x =$$

$$= (c_s^j - c_\infty) \iint\limits_{S_j} \mathbf{n}^j(\mathbf{x}) \cdot \nabla_x \phi_j(\mathbf{x})\, d\sigma_x + \sum_{k \neq j} (c_s^k - c_\infty) \iint\limits_{S_k} \mathbf{n}^k(\mathbf{x}) \cdot \nabla_x \phi_j(\mathbf{x})\, d\sigma_x,$$

where the last integral on the right-hand side vanishes since $\phi_j(\mathbf{x})$ is harmonic outside the surface S_j, whereas the first integrals on both sides can be evaluated by taking into account formulas (5) and (11), respectively, leading to the following equations ($j = 1, 2, \ldots, N$):

$$Q_j - \sum_{k \neq j} \iint\limits_{S_k} \phi_j(\mathbf{x}) \mathbf{n}^k(\mathbf{x}) \cdot \nabla_x c(\mathbf{x})\, d\sigma_x = 2\pi \mathsf{c}_j (c_s^j - c_\infty). \tag{45}$$

It should be underlined that Eq. (45) is an exact relation which, however, still contains the unknown function $c(\mathbf{x})$.

A simple approximate solution can be obtained by setting $\phi_j(\mathbf{x}) \approx \phi_j(P^k)$ for $\mathbf{x} \in S_j$, so that, in view of (5) and (43), Eq. (45) reduces to Eq. (41). This approximate solution to the multiple evaporation problem was derived by Masoud *et al.* [3] in a somewhat different way. In the context of multiple contact problems, the application of the reciprocity theorem (known as Mossakovskii's theorem [40]) dates back to Rakov [41] (see also [42]).

4.4 Kachanov's method

Returining back to the linear combination (8), we determine its coefficients by the least square residual method, i.e.,

$$\min_{d_j} \iint_{S_j} [c(\mathbf{x}) - c_{\mathrm{s}}^j]^2 \mathrm{d}\sigma_x, \tag{46}$$

where $c(\mathbf{x})$ is given by (8).

In this way, we arrive at the equations

$$\iint_{S_j} \Big(c_{\mathrm{s}}^j - c_\infty - d_j - \sum_{k\neq j} d_k \phi_k(\mathbf{x}) \Big) \mathrm{d}\sigma_x = 0, \tag{47}$$

which can be recast in the form

$$d_j + \sum_{k\neq j} \Lambda_{kj} d_k = c_{\mathrm{s}}^j - c_\infty, \tag{48}$$

where we have introduced the notation

$$\Lambda_{kj} = \frac{1}{|S_j|} \iint_{S_j} \phi_k(\mathbf{x})\, \sigma_x, \tag{49}$$

and $|S_j| = 2|S_j^+|$ is the area of the surface S_j.

Using relation (15), which links d_j and Q_j, we can rewrite Eq. (48) as follows:

$$Q_j + \sum_{k\neq j} \frac{\mathsf{C}_j}{\mathsf{C}_k} \Lambda_{kj} Q_k = 2\pi \mathsf{C}_j (c_{\mathrm{s}}^j - c_\infty). \tag{50}$$

Obviously, by setting $\phi_k(\mathbf{x}) \approx \phi_k(P^j)$ for $\mathbf{x} \in S_j$, the surface integral in (49) can be approximately evaluated as

$$\Lambda_{kj} \approx \phi_k(P^j). \tag{51}$$

Thus, in view of the previously introduced notation (43), we can write $\Lambda_{kj} \approx \lambda_{jk}$. It should be emphasized that, generally speaking, $\lambda_{jk} \neq \lambda_{kj}$.

Remark 4 According to Eq. (9), the modified Kachanov method can be reformulated in the form of minimization problem (compare with (46))

$$\min_{d_j} \iint_{S_j} [c(\mathbf{x}) - c_{\mathrm{s}}^j]^2 \frac{\partial \phi_j}{\partial n}(\mathbf{x})\, \mathrm{d}\sigma_x, \tag{52}$$

where $c(\mathbf{x})$ is given by Kachanov's approximation (8), and $j = 1, 2, \ldots, N$.

5 Evaporation of multiple thin droplets

As an example, we consider a system of N thin droplets with circular contact lines (see also [2]). Let $\varphi_j(\mathbf{x})$ denote the conductor potential of a circular infinitesimally thin disc of radius a_j, which is harmonic in the entire half-space $x_3 > 0$ and satisfies the boundary conditions $\varphi_j(x_1, x_2, 0) = 1$ for $(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2 \leq a_j^2$, and $\partial\varphi_j/\partial x_3(x_1, x_2, 0) = 0$ for $(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2 > a_j^2$. From Boussinesq's solution for the corresponding harmonic potential problem, we have

$$\varphi_j(x_1, x_2, 0) = \frac{2}{\pi} \arcsin \frac{a_j}{\sqrt{(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2}} \tag{53}$$

for $(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2 \geq a_j^2$,

$$\frac{\partial \varphi_j}{\partial x_3}(x_1, x_2, 0^+) = -\frac{2}{\pi} \frac{1}{\sqrt{a_j^2 - (x_1 - x_1^j)^2 - (x_2 - x_2^j)^2}} \tag{54}$$

for $(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2 < a_j^2$. In this case, we also have $\mathtt{c}_j = (2/\pi)a_j$.

To calculate the transmission factors, the following relation hold [26]:

$$\Lambda_{kj} = \frac{4}{\pi^2 a_j^2} \int_0^\pi \mathrm{d}\phi \int_0^{a_j} \mathcal{W}_0^{kj}(\rho, \phi)\rho\, \mathrm{d}\rho. \tag{55}$$

Here we have introduced the auxiliary notation

$$\mathcal{W}_0^{kj}(\rho, \phi) = \arcsin \frac{a_k}{\sqrt{D_{kj}^2 + \rho^2 - 2\rho D_{kj} \cos\phi}}. \tag{56}$$

From Eq. (53), we readily get

$$\lambda_{kj} = \frac{2}{\pi} \arcsin \frac{a_j}{D_{kj}}, \tag{57}$$

where $D_{kj} = |P^j - P^k|$ is the distance between the centers of the j-th and k-th droplets.

From Eqs. (13), (53), and (54), we obtain

$$\tilde{\Lambda}_{kj} = \frac{2}{\pi^2 a_j} \int_0^\pi \mathrm{d}\phi \int_0^{a_j} \mathcal{W}_0^{kj}(\rho, \phi) \frac{\rho\, \mathrm{d}\rho}{\sqrt{a_j^2 - \rho^2}}. \tag{58}$$

According to Eq. (28), we find that

$$\grave{\Lambda}_{kj} = \frac{2}{\pi} \frac{a_k}{D_{kj}}. \tag{59}$$

When $a_j = a_k$, obviously, we have $\lambda_{kj} = \lambda_{jk}$, etc.

The variation of the relative total evaporative flux from a system of two identical circular thin droplets (normalized by the non-interaction approximation) is shown in Fig. 2a. This figure reproduces the graphs shown in Figures 9(a) and 9(b) of [26], supplemented by additional curves (labeled "m") obtained using the modified Kachanov method.

In the symmetric case, the best agreement is provided by the source-function-based approximation (Section 4.2), the reciprocal-relation-based approximation (Section 4.3), and the simplified version of the Kachanov method (Section 4.4). Notably, the proposed modified Kachanov method outperforms all other methods in the asymmetric case (see Fig. 2b), which is relevant to contact problems.

Although the presented comparative analysis assesses the accuracy of the method only in a benchmarking sense, based on two representative test problems, it supports the use of the simplified Kachanov approach, which yields the same approximate solutions as the source-function and reciprocal-theorem-based methods, while being the simplest and apparently the most accurate under symmetry-type boundary conditions ($c_s^j = c_s = \text{const}$ for $j = 1, 2, \ldots, N$). A detailed comparison with existing theoretical and experimental studies on the evaporation of multiple droplets, however, lies beyond the scope of the present work.

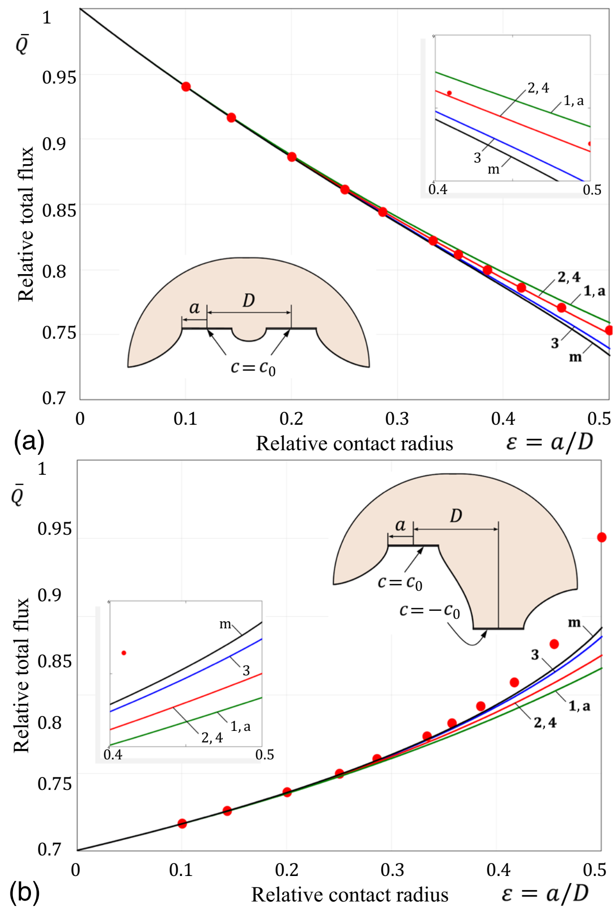

Fig. 2 Comparison of predictions of the approximate methods and the accurate analytical solution by Kobayashi [43] (red dot symbols) in the symmetric (a) and asymmetric (b) cases. Analytical models: **1** Greenwood [25]; **2** Galin (source-function-based approximation) [39] and Gladwell–Fabrikant (reciprocal-relation-based approximation) [44]; **3** Kachanov [26]; **4** simplified version of the Kachanov method [26]; **a** asymptotic model [45]; **m** proposed modified Kachanov method

6 Discussion and conclusion

First of all, the consideration of different methods for approximately solving the problem of multiple droplet evaporation is not limited to symmetric configurations or to constant Dirichlet boundary conditions. The latter generalization is described in Section 8, ‘Generalizations’ [26] (see the third point). More generally, further refinements of the methods (e.g., by including additional terms in the corresponding approximations) require solving the single-object problem for an arbitrary right-hand side of the Dirichlet boundary condition.

Also, a far-reaching analogy can be drawn between the modified Kachanov method in the form of Eq. (9) and the well-known Bubnov–Galerkin projection technique. This perspective suggests a natural generalization of the method to the solution of general multiple-object external Dirichlet problems.

Further, it is instructive to consider a simple case of spherical surfaces S_j of radii a_j $(j = 1, 2, \dots, N)$ which corresponds to the special case of hemispherical droplets (when the contact radius of a droplet equals the droplet radius). It is easy to verify that the conductor potential takes to form

$$\phi_j(\mathbf{x}) = \frac{a_j}{|\mathbf{x} - P^j|}, \quad \mathsf{c}_j = a_j; \qquad \frac{\partial \phi_j}{\partial n}(\mathbf{x}) = -\frac{a_j}{|\mathbf{x} - P^j|^2}. \tag{60}$$

Moreover, due to the mean property of harmonic functions, we will also have

$$\phi_k(P^j) = \frac{1}{|S_j|} \iint\limits_{S_j} \phi_k(\mathbf{x})\, \mathrm{d}\sigma_x, \tag{61}$$

$$\iint\limits_{S_j} \phi_k(\mathbf{x}) \frac{\partial \phi_j}{\partial n}(\mathbf{x})\, \mathrm{d}\sigma_x = -4\pi a_j \phi_k(P^j). \tag{62}$$

Hence, in light of (60)–(62), formulas (13), (28), (43), and (49) yield, respectively,

$$\tilde{\Lambda}_{kj} = \phi_k(P^j) = \frac{a_k}{D_{jk}}, \quad \grave{\Lambda}_{kj} = \frac{a_k}{D_{jk}}, \tag{63}$$

$$\lambda_{kj} = \phi_j(P^k) = \frac{a_j}{D_{jk}}, \quad \Lambda_{kj} = \phi_k(P^j) = \frac{a_k}{D_{jk}}, \tag{64}$$

from where it immediately follows that

$$\frac{\mathsf{c}_j}{\mathsf{c}_k} \tilde{\Lambda}_{kj} = \frac{\mathsf{c}_j}{\mathsf{c}_k} \Lambda_{kj} = \lambda_{kj} = \frac{a_j}{D_{jk}}, \tag{65}$$

and therefore, in the special case under consideration, all the examined approximate solutions, including the asymptotic model (27), completely agree.

It is evident that the modified Kachanov method (especially within the framework of the multiple-evaporation problem, as compared with the multiple-contact problem) is computationally more expensive than the original Kachanov method, not to mention the source-function-based and reciprocal-relation-based approximations. As shown by the example of two circular thin droplets (see Fig. 2), it is not *a priori* possible to determine which of the considered methods is more accurate, except for the asymptotic modeling approach (see Section 4.1), utilizing only the leading asymptotic term of the conductor potential (see the asymptotic formula (24)). Nevertheless, the accuracy of the different approximations for closely spaced droplets remains to be investigated, for example, numerically for various cluster configurations, particularly for droplets of different sizes.

To conclude, this paper is dedicated to the 80th anniversary of Mark Kachanov. The idea of the modified Kachanov method (in the form presented in Remark 4) had crystallized as early as July 2020, in the first complete draft of our joint paper [26]. Work on that paper began in July 2008, when the concept of extending the Kachanov method to frictionless contact problems was first developed in principle.

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