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Author for correspondence:
Samuele Sottile
e-mails: samuele.sottile@mau.se; samuele.sottile@math.lu.se

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Inverse spectral Love problem via Weyl–Titchmarsh function

Samuele Sottile[1,2]

[1]Centre for Mathematical Sciences, Lund University, Lund 221 00, Sweden
[2]Department of Materials Science and Applied Mathematics, Malmö University, Malmö SE-205 06, Sweden

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In this paper, we prove an inverse resonance theorem for the half-solid with vanishing stresses on the surface via Weyl–Titchmarsh function. Using a semi-classical approach, it is possible to simplify this three-dimensional problem of the elastic wave equation for the half-solid as a Schrödinger equation with Robin boundary conditions on the half-line. The goal of the paper is to establish a method to recover the potential from eigenvalues and resonances through the Weyl–Titchmarsh function for non-self-adjoint problems and to establish a one-to-one and onto map between suitable function spaces. Moreover, we produce an algorithm in order to retrieve the shear modulus from the eigenvalues and resonances, via the spectral data.

1. Introduction

(a) Inverse resonance problems, previous results

The Love boundary value problem for the vertically inhomogeneous elastic isotropic medium in the half-space (see [1,2]):

$$-\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_2}{\partial Z} + \hat{\mu}\,|\,\xi\,|^2\varphi_2 = \omega^2\varphi_2, \tag{1.1}$$

$$\frac{\partial \varphi_2}{\partial Z}(0) = 0, \tag{1.2}$$

where $Z \in (-\infty, 0]$ is the coordinate with direction normal to the boundary, $\hat{\mu}$ is the density-normalized shear modulus, ξ is the dual of the coordinate vector (x, y) parallel to the boundary, ω is the frequency and φ_2 is the component of the displacement vector on the y direction. Equation (1.1) describes the motion of an infinitesimal element of elastic solid on a direction lying on a plane parallel to the Earth's surface. The boundary condition (1.2) says that the infinitesimal element has zero normal velocity on the Earth's surface, which is line with the fact that Love waves are transverse waves. Equations (1.1), (1.2) are obtained in Nakamura *et al.* [1] after decoupling the elastic wave equation for infinitesimal solid and using the semiclassical limit. A change of variable (see §1.(b)) in equations (1.1), (1.2) leads to a Schrödinger equation:

$$-u'' + Vu = k^2u, \tag{1.3}$$

with Robin boundary condition:

$$u'(0) + hu(0) = 0. \tag{1.4}$$

Equations (1.1), (1.2) describe the equation of motion of Love seismic surface waves. Surface waves are waves that travel close to the Earth's surface with amplitude decaying exponentially with the distance to the Earth's surface. Love waves are a type of seismic surface waves (the other type is the Rayleigh waves) which are transverse, in the sense that the oscillation of an infinitesimal element of elastic solid is perpendicular to the direction of propagation of the wave. The boundary condition (1.2) is in line with the Love waves being transverse waves, hence not implying oscillations on the vertical axis.

A first complete result on the inverse resonance problem for Love seismic surface waves was obtained in Sottile [2]. In Sottile [2], the author reconstructs uniquely the potential of a Schrödinger equation with Robin boundary condition from eigenvalues and resonances, partly using previous results of Marchenko [3], employing a different class of potential and different boundary condition, and Korotyaev [4], employing the same class of potential but different boundary condition. In this paper, we propose an alternative way of reconstructing the potential (and shear modulus) via Weyl–Titchmarsh function formalism, that can be generalized to non-self-adjoint problems. This approach is inspired by the paper [5], which uses a similar approach for the Rayleigh problem, consisting of a non-self-adjoint Schrödinger (matrix-valued) equation with Robin boundary condition. The same is reproduced later by de Hoop & Iantchenko [6] for the same Rayleigh system as in Beals *et al*. [5]. Some other remarkable papers about surface (Rayleigh) waves are the paper from Pekeris (see [7]), who obtained uniqueness of density and Lame parameters through the knowledge of the displacements at the boundary and under the hypothesis of them being real analytic; or a paper from Markushevich (see [8]) that obtained uniqueness of the reconstruction of the smooth potential through known boundary values of solutions of the problem for two given independent sources. In de Hoop & Iantchenko [9], they find direct results on location of the resonances for the Rayleigh system.

In relation to seismology, solving the inverse Love problem means reconstructing the parameters that determine the elasticity of the medium in the interior of the Earth from measurements performed on the boundary of the Earth's surface, which are, for example, the frequencies or the wave numbers of surface waves (eigenvalues and resonances). The Earth is a compact domain, but, for simplification, we consider it as a flat half space $\mathbb{R}^2 \times (-\infty, 0]$.

There are only a few examples of complete characterizations of inverse resonance problems, for instance by Korotyaev (see [4]), who solved it on the half-line for compactly supported potentials with Dirichlet boundary condition, or Christiansen, who solved it on the whole line for step-like potentials (see [10]), using some results from an earlier paper of Cohen–Kappeler [11]. Some other examples are [12–14]. Gesztesy [15] shows a characterization result for real integrable potentials in the Schrödinger operator on the half line using the Krein spectral shift function, which is connected to the scattering phase $\delta(k)$ through the Birman–Krein formula. In Iantchenko & Korotyaev [16], asymptotical values of resonances are obtained for the periodic Jacobi operator with finitely supported perturbations. There are some other examples of inverse problems in seismology. For example, in de Hoop & Iantchenko [6], they analyse an inverse spectral problem for the semi-classical Rayleigh operator starting with spectral data being the Weyl–Titchmarsh matrix.

(b) Semi-classical description of the Love waves system

Starting from equations (1.1), (1.2), we perform a transformation so that the resulting boundary value problem assumes a Schrödinger-type form. Once we have performed the calibration transform, we need to solve an inverse resonance Schrödinger problem with Robin boundary condition, where the eigenvalues and resonances are the poles of the resolvent with respect to the parameter k, respectively for $\operatorname{Im} k > 0$ and $\operatorname{Im} k \leq 0$.

The main goal of this paper is to retrieve the shear modulus $\hat{\mu}$ ($\hat{\mu} = \mu/\rho$, with ρ being the density) as we explain in remark 3.5. This is obtained by an application of a characterization (see theorem 3.3) between a class $\mathbb{M}_{x_I}$ of Weyl functions (see definition 3.1) and a class $\mathbb{V}^1_{x_I}$ of potentials (see definition 1.5).

We make a simplifying assumption in the following.

Assumption 1.1. (Homogeneity). *We assume that below a certain depth Z_I, the medium is homogeneous, so the shear modulus is constant*:

$$\hat{\mu}(Z) = \hat{\mu}(Z_I) := \hat{\mu}_I \qquad \text{for} \quad Z \leq Z_I. \tag{1.5}$$

We perform the calibration transform in equations (1.1), (1.2) as

$$\varphi_2 = \frac{1}{\sqrt{\hat{\mu}}}u, \quad \frac{d}{dZ}\left(\hat{\mu}\frac{d}{dZ}\varphi_2\right) = \frac{1}{4}\hat{\mu}^{-\frac{3}{2}}(\hat{\mu}')^2 u - \frac{1}{2}\hat{\mu}^{-\frac{1}{2}}\hat{\mu}'' u + \hat{\mu}^{\frac{1}{2}} u'',$$

and we get:

$$u'' - |\xi|^2 u = \left[\frac{1}{2}\frac{\hat{\mu}''}{\hat{\mu}} - \frac{1}{4}\left(\frac{\hat{\mu}'}{\hat{\mu}}\right)^2 - \frac{1}{\hat{\mu}}\omega^2\right] u.$$

We set the quasi momentum $k := \sqrt{\frac{\omega^2}{\hat{\mu}_I} - |\xi|^2}$ and:

$$V = \frac{1}{2}\frac{\hat{\mu}''}{\hat{\mu}} - \frac{1}{4}\left(\frac{\hat{\mu}'}{\hat{\mu}}\right)^2 - \frac{1}{\hat{\mu}}\omega^2 + \frac{1}{\hat{\mu}_I}\omega^2 = \frac{(\sqrt{\hat{\mu}})''}{\sqrt{\hat{\mu}}} - \frac{1}{\hat{\mu}}\omega^2 + \frac{1}{\hat{\mu}_I}\omega^2, \tag{1.6}$$

where $\hat{\mu}_I := \hat{\mu}(Z_I)$ is the value of the shear modulus at the depth Z_I, below which the medium is homogeneous. By assumption 1.1, $\hat{\mu}(Z) = \hat{\mu}_I$ constant for $Z \leq Z_I$, hence also the derivatives $\hat{\mu}'$ and $\hat{\mu}''$ vanish for $Z \leq Z_I$. This implies that the potential V has compact support and depends only on Z as we fixed ω and let our spectral parameter ξ vary. In this way, the potential $V = V_\omega$ can be parametrized by ω and we let ξ vary.

Remark 1.2. *Usually for the Love problem, there are two different types of transform to reduce the Sturm–Liouville equation to the Schrödinger equation: the calibration transform and the Liouville transform. The Liouville transform would lead to a Schrödinger-type equation with potential depending on the wave vector ξ and with the frequency ω being the spectral parameter; while in the calibration transform, we would get a Schrödinger equation with potential depending on ω and with ξ being the spectral parameter. In this paper, we use the calibration transform, and study eigenvalues and resonances in terms of ξ, as we will do in a forthcoming paper about the Rayleigh operator, where the only possible reduction is with ξ being the spectral parameter.*

Remark 1.3. *We will assume that the potential $V \in \mathbb{V}^1_{x_I}$ (definition 1.5), that implies the shear modulus $\hat{\mu}$ to be constant below the depth Z_I and to be different than $\hat{\mu}_I$ in an interval of type* $(Z_I, a + Z_I)$ for $a > 0$.

The Love scalar equation takes the following form:

$$-u'' + Vu = \lambda u, \qquad \lambda = k^2, \tag{1.7}$$

with corresponding boundary condition that becomes of Robin type after the transformation:

$$u'(0) + hu(0) = 0, \qquad h = -\frac{1}{2}\frac{\hat{\mu}'(0)}{\hat{\mu}(0)}. \tag{1.8}$$

Remark 1.4. *Here and throughout, we consider h given, as it is determined by the values of the density-normalized shear modulus and its derivative at the Earth's surface, which are always measurable. However, one could also consider h unknown and recover it from* (2.8).

To resemble the classical formulation, we make the substitution $Z = -x$, which leads the domain to become $[0, +\infty)$ and we study the problem in terms of k. In our case, the potential of the Schrödinger operator is real because we are considering an elastic medium. In the case of an inelastic medium, we would have a complex potential that implies the loss of part of the energy which is converted into heat. We make a self-adjoint realization in $L^2(\mathbb{R}_+)$ of the operator in (1.7) due to the boundary condition (see [17]). Then, the operator appearing on the left-hand side of (1.7) prescribed with the domain:

$$D = \{u \in H^2[0, +\infty): Vu \in L^2[0, +\infty), u'(0) + hu(0) = 0\}, \tag{1.9}$$

and the L^2 inner product is self-adjoint.

(c) Preliminary results

In this subsection, we list some preliminary results which are needed throughout the whole paper (see [2]). First, we define the class $\mathbb{V}^1_{x_I}$, to which the potential belongs throughout the whole paper.

Definition 1.5. *We denote by* $\mathbb{V}^1_{x_I}$ *the class of real potentials* V *such that* $V, V' \in L^1(\mathbb{R}_+)$, $\operatorname{supp}V \subset [0, x_I]$ *for some* $x_I > 0$ *and for each* $\epsilon > 0$, *the set* $(x_I - \epsilon, x_I) \cap \operatorname{supp}V$ *has positive Lebesgue measure*.

Below, we define the solution to (1.7) satisfying the radiation condition.

Definition 1.6. (Jost solution). *The* ***Jost solutions*** $f^{\pm}$ *are the unique solutions to the differential equation (1.7) that satisfy the following condition*:

$$f^{\pm}(x, k) = e^{\pm ikx} \qquad \text{for} \quad x > x_I. \tag{1.10}$$

The Jost solution satisfies the Volterra-type equation:

$$f(x, t) = e^{ikx} - \int_x^{\infty} \frac{\sin[k(x-t)]}{k} V(t) f(t, k) dt. \tag{1.11}$$

It is a known that the spectrum for the operator (1.7) with domain (1.9) consists of a finite number of purely imaginary and simple eigenvalues in k, there are no real eigenvalues and the eigenfunctions are real (see [2,18]).

From the values of the Jost solution at the boundary we can define the Jost function as follows.

Definition 1.7. (Jost function). *We define the* ***Jost function*** $f_h(k)$ *of the Schrödinger operator* $-\frac{d^2}{dx^2} + V$ *in* (1.7) *with Robin boundary condition* (1.8) *as the quantity*:

$$f_h(k) = f(0, k)h + f'(0, k) \tag{1.12}$$

where $f(0, k)$ *is the Jost solution evaluated at* $x = 0$.

Below, we define the regular solution (see [6, section 1.(b)]).

Definition 1.8. (Regular solution). *We define the regular solution* φ *of the Cauchy problem* (1.7) *with Robin boundary condition* (1.8) *as the quantity*:

$$\varphi(x, k) = -\frac{1}{2ik}\left[\overline{f_h(k)}\, f(x, k) - f_h(k)\, \overline{f(x, k)}\right]. \tag{1.13}$$

We define the eigenvalues as the zeros of the Jost function f_h in the physical sheet Im $k > 0$ and we denote as (scattering) resonances the zeros of f_h in the unphysical sheet Im $k < 0$. Eigenvalues lead to a L^2 solution of the differential equation, while resonances lead to a non-L^2 solution. We jointly enumerate the zeros of f_h, which are eigenvalues and resonances, as $(k_j)_{j \in \mathbb{N}}$, where $k_1, \cdots, k_N$ are the eigenvalues.

We recall a result of analyticity of Jost solution and Jost function for potential in the class $\mathbb{V}^1_{x_I}$ and some estimates of those. For more details and proofs, see Sottile [2].

Theorem 1.9. *For each fixed $x \geq 0$, the Jost solution $f(x,k)$ and the Jost function $f_h(k)$ are entire in k.*

Lemma 1.10. (Uniform bounds on Jost function). *Let $V \in \mathbb{V}^1_{x_I}$. Then the Jost function is of exponential type and satisfies the following estimates:*

$$|f(0,k) - 1| \leq e^{(|\mathrm{Im}k| - \mathrm{Im}k)x_I} a e^a \tag{1.14}$$

$$\left| f(0,k) - 1 + \frac{\hat{V}(0) - \hat{V}(k)}{2ik} \right| \leq \frac{a^2}{2} e^{(|\mathrm{Im}k| - \mathrm{Im}k)x_I} e^a \tag{1.15}$$

$$|f_h(k) - ik| \leq \|V\| e^{(|\mathrm{Im}k| - \mathrm{Im}k)x_I} e^a \tag{1.16}$$

$$\left| f_h(k) - ik - h + \frac{\hat{V}(0) + \hat{V}(k)}{2} \right| \leq \left[|h| + \frac{\|V\|}{2} \right] a e^{(|\mathrm{Im}k| - \mathrm{Im}k)x_I} e^a, \tag{1.17}$$

where $\hat{V}(k) = \int_0^{x_I} e^{2ikt} V(t) dt$ is the Fourier transform of the potential V and $a = \frac{||V||}{\max(1, |k|)}$, with $||V|| := \int_{\mathbb{R}} |V(x)| dx$.

Lemma 1.11. *If $V \in \mathbb{V}^1_{x_I}$ and if $k_1, \ldots, k_N \in i\mathbb{R}_+$ are the zeros of the Jost function $f_h(k)$ such that $|k_1| > \ldots > |k_N| > 0$, then the following inequalities hold:*

$$i(-1)^j \dot{f}_h(k_j) > 0, \quad \text{and} \quad (-1)^j f_h(-k_j) < 0, \qquad \text{for } j = 1, \ldots, N, \tag{1.18}$$

where the dot denotes the derivative with respect to k.

2. The spectral problem

In this section, we introduce the Weyl function formalism and we recover a Gelfand–Levitan-type equation (see proposition 2.4) following a similar procedure as in Novikov *et al*. [19, chapter 1, section 1] and Beals *et al*. [5] adapted to our Love scalar boundary value problem. Then, we establish a bijection (see theorem 3.3) between a class $\mathbb{M}_{x_I}$ of Weyl function (see definition 3.1) and the class $\mathbb{V}^1_{x_I}$ (see definition 1.5). We do not follow the usual approach in which the Weyl–Titchmarsh function is defined to be a Herglotz–Nevanlinna function and from which, by using its integral representation and the Stieltjes inversion formula, one can obtain the spectral measure (see [20, theorem 9.17]). Instead, we follow the approach of [21, chapter 2] and define the Weyl function in a different way (see definition 2.2), which is more suitable for non-self-adjoint problems.

(a) Estimates of the regular solution

We want to obtain an estimate for the regular solution φ in the limit $|k| \to \infty$. We start from the Volterra-type expression for the regular solution φ

$$\varphi(x,k) = \cos kx - h\frac{\sin kx}{k} + \int_0^x \frac{\sin[k(x-t)]}{k} V(t)\varphi(t,k)\,dt\,. \tag{2.1}$$

We can easily see that this function satisfies the differential equation and the boundary condition. Indeed

$$\varphi'(x,k) = -k\sin kx - h\cos kx + \int_0^x \cos[k(x-t)]V(t)\varphi(t,k)dt \tag{2.2}$$

and

$$\varphi''(x,k) = -k^2\cos kx + hk\sin kx + V(x)\varphi(x,k) - \int_0^x k\sin[k(x-t)]V(t)\varphi(t,k)dt;$$

thus $-\varphi''(x,k) + V(x)\varphi(x,k) = k^2\varphi(x,k)$. Moreover,

$$\varphi'(0,k) = -h, \qquad \varphi(0,k) = 1,$$

so also the boundary condition $\varphi'(0,k) + h\varphi(0,k) = 0$ is satisfied. Taking the absolute value of (2.1) and since $|\sin kx| \le \exp(|\eta|x)$ and $|\cos kx| \le \exp(|\eta|x)$, where $\eta = \mathrm{Im}\,k$, we get:

$$|\varphi(x,k)| \le \exp(|\eta|x) + \frac{\exp(|\eta|x)}{|k|} + \int_0^x \frac{\exp(|\eta|(x-t))}{|k|}|V(t)||f(t,k)|dt.$$

We define $\beta_T(k) = \max_{0\le x\le T}(|\varphi(x,k)|)\exp(-|\eta|x)$ and we have then for $|k| > 1$

$$\beta_T(k) \le C_1 + \frac{1}{|k|}\beta_T(k)\int_0^T |V(t)|\,dt \le C_1 + \frac{1}{|k|}\beta_T(k)\int_0^\infty |V(t)|\,dt,$$

which for $|k| \to \infty$ implies $\beta_T(k) = O(1)$, hence $\varphi(x,k) = O(\exp(|\eta|x))$. Substituting this estimate on (2.1), we get $|\varphi(x,k)| \le C\exp(|\eta|x)$. Doing the same for the derivative of $\varphi(x,k)$, as in (2.2), we get:

$$|\varphi^{(\nu)}(x,k)| \le C|k|^\nu\exp(|\eta|x), \qquad \nu = 0,1, \qquad |k| \gg 1 \tag{2.3}$$

uniformly in x.

(b) Properties of Weyl function

In this subsection, we will define the Weyl solution and the Weyl function and present their properties. These quantities enable another approach to solve the inverse problem (see [21], section 2.2]) and will also enable us to recover the Gelfand–Levitan equation in an alternative way (see §2.(c)), as the Gelfand–Levitan equation is usually recovered from the spectral measure.

In this subsection, λ and k are always related via $\lambda = k^2$ defined initially for Im $k > 0$. Below, we give a definition of Weyl solution that uses those of the Jost solution (definition 1.6) and the Jost function (definition 1.7) given in the previous section.

Definition 2.1. (Weyl solution). *We define the* ***Weyl solution*** $\phi(x,\lambda)$ *as the function*:

$$\phi(x,\lambda) = \frac{f(x,k)}{f_h(k)}, \qquad \mathrm{Im}\ k > 0. \tag{2.4}$$

This function satisfies the differential equation $-\phi'' + V\phi = \lambda\phi$ because the Jost solution does, but does not satisfy the Robin Boundary condition. In particular:

$$\phi'(0,\lambda) + h\phi(0,\lambda) = 1 \tag{2.5}$$

$$\phi(x,\lambda) = O(e^{ikx}) \qquad x\to\infty, \quad k\in\Sigma, \tag{2.6}$$

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where we define the set $\Sigma := \{k \in \mathbb{C} : \text{Im } k \geq 0, k \neq 0\}$. From (1.14) and (1.17) in lemma 1.10, we get the asymptotics on the Weyl solution for large k:

$$\phi^{(\nu)}(x,\lambda) = (ik)^{\nu-1}\exp(ikx)\left(1 + o\left(\frac{1}{k}\right)\right), \qquad \nu = 0,1, \quad |k| \to \infty. \tag{2.7}$$

The Weyl solution is uniquely determined by the differential equation $-\phi'' + V\phi = \lambda\phi$ (see (1.7)) and the boundary condition (2.5).

Definition 2.2. (Weyl function). *We define the* ***Weyl function*** $M(\lambda)$ *(or Weyl–Titchmarsh function) as the function*:

$$M(\lambda) := \phi(0,\lambda) = \frac{f(0,k)}{f_h(k)}, \qquad \lambda = k^2, \text{Im } k > 0.$$

Remark 2.3. *In some other textbooks, the Weyl–Titchmarsh function for Robin boundary condition is defined as* $M(\lambda) = \frac{hf'(0,k) - f(0,k)}{f_h(k)}$ *(see [20, formula 9.52, chapter 9]). This choice entails that* $M(\lambda)$ *is* a Herglotz–Nevanlinna function and using its properties, it is possible to obtain the spectral measure (see [20, theorem 9.17]). *In our treatment, we follow the approach and the definition of [11, definition 2.1.69].*

Remark 2.4. *The zeros of the Jost function (definition 1.7) correspond to the poles of the Weyl function (see theorem 2.15 below). Indeed, at zeros* $k = k_j$ *of the Jost function,* $f(0,k_j) = -\frac{1}{h}f'(0,k_j) \neq 0$.

Remark 2.5. *The Weyl function* $M(\lambda)$ *maps* $f_h(k)$ *to* $f(0,k)$*, so* M *is the Robin-to-Dirichlet map, since the Jost function in the Dirichlet boundary value problem* ($h = \infty$) *is precisely* $f(0,k)$ *(see* [4]*). In the case of Dirichlet boundary condition, the Weyl function is usually defined as* $\frac{f'(0,k)}{f(0,k)}$ *(see* [20, *formula 9.52, chapter 9]) which is Herglotz–Nevanlinna and can be reconstructed by the Dirichlet and Neumann eigenvalues and resonances.*

From the asymptotics of the Jost solution and Jost function, we obtain the asymptotics of the Weyl function, as described in the following lemma.

Lemma 2.6. *Let* $V \in \mathbb{V}^1_{x_I}$*, then the Weyl function (see definition 2.2) has the asymptotic expansion:*

$$M(\lambda) = \frac{1}{ik}\left[1 - \frac{h}{ik} + \frac{\hat{V}(k)}{ik} + o(k^{-1})\right], \qquad |k| \to +\infty. \tag{2.8}$$

Proof. From (1.17) and (1.14) in lemma 1.10, we can get the asymptotics of the Weyl function:

$$M(\lambda) = \frac{1}{ik}\left(1 + O\left(\frac{1}{k}\right)\right), \qquad |k| \to +\infty.$$

Using (1.17) and (1.15), we can get higher order terms of the expansion of the Weyl function in terms of k:

$$\begin{aligned}
M(\lambda) &= \left(1 - \frac{\hat{V}(0) - \hat{V}(k)}{2ik} + o(k^{-1})\right)\left(\frac{1}{ik + h - \frac{\hat{V}(0)+\hat{V}(k)}{2} + o(1)}\right) \\
&= \frac{1}{ik}\left(1 - \frac{\hat{V}(0) - \hat{V}(k)}{2ik} + o(k^{-1})\right)\left(1 - \frac{h}{ik} + \frac{\hat{V}(0) + \hat{V}(k)}{2ik} + o(k^{-1})\right) \\
&= \frac{1}{ik}\left[1 - \frac{h}{ik} + \frac{\hat{V}(k)}{ik} + o(k^{-1})\right].
\end{aligned}$$

∎

Note that we can write the Weyl solution as:

$$\phi(x,\lambda) = \theta(x,k) + M(\lambda)\varphi(x,k), \tag{2.9}$$

where $\varphi(x,k)$ and $\theta(x,k)$ are solutions of (1.7) satisfying:

$$\begin{aligned} \theta(0,k) &= 0 \qquad \theta'(0,k) = 1 \\ \varphi(0,k) &= 1 \qquad \varphi'(0,k) = -h, \end{aligned} \tag{2.10}$$

and $\varphi(x,k)$ is the regular solution as in definition 1.8. We can see that

$$W(\varphi(x,k), \phi(x,\lambda)) = W(\varphi(x,k), \theta(x,k)) = 1. \tag{2.11}$$

We denote:

$$\Lambda = \left\{\lambda = k^2 : k \in \Sigma,\ f_h(k) = 0\right\}$$

and

$$\Lambda' = \left\{\lambda = k^2 : \operatorname{Im} k > 0,\ f_h(k) = 0\right\}.$$

The set Λ' consists of all the eigenvalues of the differential equation $-f'' + Vf = \lambda f$ (see (1.7)). By lemma 2.6, the Weyl function at the second order can also be written as:

$$M(\lambda) = \frac{1}{ik}\left(1 - \frac{h}{ik} + \frac{1}{ik}\int_0^{\infty} V(t)e^{2ikt}dt + o\left(\frac{1}{k}\right)\right), \qquad |k| \to +\infty, k \in \Sigma. \tag{2.12}$$

The following definition of the domain of λ comes from [21, chapter 2].

Definition 2.7. *We define Π as the λ-plane with the cut $\lambda \geq 0$, and $\Pi_1 = \overline{\Pi}\setminus\{0\}$. Π and Π_1 must be considered as a subset of the Riemann surface of the square-root function*.

In definition 2.7, we stated that Π and Π_1 must be considered as a subset of the Riemann surface of the square root because λ as a square of k ($k = \sqrt{\lambda}$) lays in two copies of the complex plane with cuts on the positive real axis and glued together. Hence, Π and Π_1 live in the first (physical sheet) of these two sheets. Since the cut is placed in the real positive axis of the $\Pi\lambda$-plane, the Weyl function has a jump between above and below the cut. This motivates the following definition.

Definition 2.8. (Jumps of Weyl function). *We define*

$$T(\lambda) = \frac{1}{2\pi i}\left(M^-(\lambda) - M^+(\lambda)\right), \qquad \lambda > 0, \tag{2.13}$$

to be the jumps of the Weyl function $M(\lambda)$ (see definition 2.2), where

$$M^{\pm}(\lambda) = \lim_{z \to 0, \operatorname{Re} z > 0} M(\lambda \pm iz).$$

From Definition 2.13, we can see that $T(\lambda)$ represents the jumps (discontinuity points of the first kind) of the Weyl function. Thanks to (2.12) and (2.13), we get the following expansion for $T(\lambda)$:

$$\begin{aligned} T(\lambda) &= \frac{1}{2i\pi k}\left[-\frac{1}{ik}\left(2 + \frac{1}{ik}\int_0^{\infty} V(t)\left(e^{2ikt} - e^{-2ikt}\right)dt + o\left(\frac{1}{k}\right)\right)\right] \\ &= \frac{1}{\pi k}\left(1 + \frac{1}{k}\int_0^{\infty} V(t)\sin 2kt\, dt + o\left(\frac{1}{k}\right)\right), \qquad k \to +\infty. \end{aligned}$$

For $a > 0$, we consider the points $\lambda = a \pm i0$ in Π_1. For $\lambda = k^2$, the point $\lambda = a + i0 \in \Pi_1$ corresponds to $k = \sqrt{a + i0} > 0$ in the positive real axis of the k complex plane, while $\lambda = a - i0 \in \Pi_1$ corresponds to the point $k = \sqrt{a - i0} < 0$ situated on the negative real axis for k.

Definition 2.9. (Spectral normalizing constant). *We define the spectral normalizing constant α_j to be the complex numbers*:

$$\alpha_j := \operatorname{Res}_{\lambda = \lambda_j} M(\lambda), \qquad j = 1, \ldots, N$$

where $\{\lambda_j\}_{j=1}^{N} = \Lambda'$.

In the following proposition, we connect the jump $T(\lambda)$ of the Weyl function to the Jost function.

Proposition 2.10. *Let $T(\lambda)$ be the jumps of the Weyl function as in definition 2.8, then:*

$$T(\lambda) = \frac{k}{\pi |f_h(k)|^2}, \qquad k > 0. \tag{2.14}$$

Proof. We follow the argument in Freiling and Yurko [21, page 134]). Identity (2.14) holds if the following identities are true:

$$W(f(x,k), \overline{f(x,k)}) = -2ik \tag{2.15}$$

$$\overline{f(x,k)} = f(x,-k), \qquad \overline{f_h(k)} = f_h(-k). \tag{2.16}$$

Those identities hold as the problem (1.7), (1.8) with domain (1.9) is self-adjoint. Indeed, $k^2 - iz$ is a complex number with real part $\mathrm{Re}(k^2) + \mathrm{Im}\ z$ and imaginary part equal to $\mathrm{Im}(k^2) - \mathrm{Re}\ z$. This complex number in the k complex plane corresponds to the roots $|k_z|e^{i\theta_z}$ and $|k_z|e^{i(\theta_z+\pi)}$, where

$$|k_z| = \left((\mathrm{Re}(k^2) + \mathrm{Im}\ z)^2 + (\mathrm{Im}(k^2) - \mathrm{Re}\ z)^2\right)^{1/4}$$

$$\theta_z = \arctan\left(\frac{\mathrm{Im}(k^2) - \mathrm{Re}\ z}{2(\mathrm{Re}(k^2) + \mathrm{Im}\ z)}\right).$$

In the limit $z \to 0$ along $z > 0$, these two solutions converge to k and $-k$, respectively. Hence, we have $M^-(\lambda) = \frac{f(0,-k)}{f_h(-k)}$ and:

$$\begin{aligned} T(\lambda) &= \frac{1}{2\pi i}\left(\frac{f(0,-k)}{f_h(-k)} - \frac{f(0,k)}{f_h(k)}\right) = \frac{1}{2\pi i}\left(\frac{\overline{f(0,k)}}{\overline{f_h(k)}} - \frac{f(0,k)}{f_h(k)}\right) \\ &= \frac{1}{2\pi i}\left(\frac{\overline{f(0,k)}(f'(0,k) + hf(0,k)) - f(0,k)(\overline{f'(0,k)} + h\overline{f(0,k)})}{|f_h(k)|^2}\right) \\ &= \frac{1}{2\pi i}\left(\frac{W(\bar{f}, f)}{|f_h(k)|^2}\right) = \frac{k}{\pi |f_h(k)|^2}, \end{aligned}$$

where in the second step, we used (2.16) and in the last, we used (2.15). ∎

From the previous proposition, we can see that we can recover the jump function from the Jost function, but not the converse. The eigenvalues $\{k_n\}_{n=1}^N$, the spectral norming constants $\{\alpha_n\}_{n=1}^N$ and the jump function $T(\lambda)$ are usually considered in the literature as the data for the inverse spectral problem (see [21, definition 2.3.1]).

The following results are useful for the inverse result at the end of this paper.

Lemma 2.11. *The following holds*:

$$\frac{k}{f_h(k)} = O(1), \qquad k \to 0,\ \mathrm{Im}\, k \geq 0. \tag{2.17}$$

Proof. We follow the proof of Cohen & Kappeler [11, theorem 2.3.5]. Since $W(f(x,k), f(x,-k)) = -2ik$ and $f_h(k) = f'(x,k) + hf(x,k)$, we have that:

$$\begin{aligned} -2ik &= f(0,k)f'(0,-k) - f'(0,k)f(0,-k) = f(0,k)(f_h(-k) - hf(0,k)) \\ &\quad -(f_h(k) + hf(0,k))f(0,-k) = f(0,k)f_h(-k) - f_h(k)f(0,-k). \end{aligned}$$

We set:

$$g(k) = \frac{2ik}{f_h(k)}$$

so, for real $k \neq 0$, we have

$$g(k) = f(0, -k) + S(k)f(0, k),$$

where $S(k) = -f_h(-k)/f_h(k)$ is the scattering function. Because of the property $\overline{f_h(k)} = f_h(-k)$, we know that $f_h(k)$ and $f_h(-k)$ have the same modulus, so $|S(k)| = 1$ for real $k \neq 0$. Let $\lambda_j = k_j^2$, $k_j = i\tau_j$, $0 < \tau_1 < \ldots < \tau_m$ and denote Σ_{τ^*} as

$$\Sigma_{\tau^*} = \{k : \mathrm{Im}\, k > 0, \ |k| < \tau^*\},$$

where $\tau^* = \tau_1/2$, considering the values k_j corresponding to the eigenvalues λ_j ordered from the smallest to the largest. The function $g(k)$ is analytic in Σ_{τ^*} and continuous in $\bar{\Sigma}_{\tau^*} \setminus \{0\}$ and from the estimates on the Jost solution we can say that:

$$|g(k)| \leq C \qquad \text{for real} \quad k \neq 0.$$

With this last estimate, we see that $g(k)$ has a removable singularity in the origin, and consequently $g(k)$ is continuous in Σ_{τ^*} and (2.17) is satisfied. ∎

Proposition 2.12. *The spectral normalizing constants α_j from definition 2.9 are strictly positive and are given by:*

$$\alpha_j = 4k_j^2 \left[\frac{-i}{f_h(-k_j)\dot{f}_h(k_j)} \right] > 0. \tag{2.18}$$

Proof. We recall the regular solution:

$$\varphi(x, k) = -\frac{1}{2ik}[f_h(-k)f(x, k) - f_h(k)f(x, -k)]$$

that, when k_j is a zero of the Jost function, becomes the eigenfunction:

$$\varphi(x, k_j) = -\frac{1}{2ik_j}[f_h(-k_j)f(x, k_j)].$$

We know that $\varphi(x, k)$ satisfies $\varphi(0, k) = 1$ (see (2.10)), hence:

$$-2ik_j = f_h(-k_j)f(0, k_j). \tag{2.19}$$

From the definition of α_j, we can write:

$$\alpha_j = \mathrm{Res}_{\lambda = \lambda_j} M(\lambda) = \lim_{\lambda \to \lambda_j} \frac{(\lambda - \lambda_j)f(0, k)}{(k - k_j)\frac{d}{dk}f_h(k)} = \frac{2k_j f(0, k_j)}{\frac{d}{dk}f_h(k)|_{k = k_j}}. \tag{2.20}$$

Plugging in (2.19), we get

$$\alpha_j = \frac{2k_j(-2ik_j)}{f_h(-k_j)\frac{d}{dk}f_h(k)|_{k = k_j}} = -\frac{4ik_j^2}{f_h(-k_j)\dot{f}_h(k_j)} = 4k_j^2\left[\frac{-i}{f_h(-k_j)\dot{f}_h(k_j)}\right] > 0,$$

where the last inequality follows from (1.18) in lemma 1.11 and k_j^2 being negative. ∎

The following theorem shows a representation formula for the Weyl function $M(\lambda)$, which can be reconstructed from the jumps $T(\lambda)$, the spectral normalizing constants α_j and the eigenvalues λ_j, as in Freiling and Yurko [21, lemma 2.3.1].

Theorem 2.13. *The Weyl function is uniquely determined by the specification of the spectral data $(T(\lambda), \{\lambda_k, \alpha_k\}_{k=1}^N)$ via the formula:*

$$M(\lambda) = \int_0^\infty \frac{T(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^N \frac{\alpha_k}{\lambda - \lambda_k}, \quad \lambda \in \Pi \setminus \Lambda'. \tag{2.21}$$

Proof. We follow the proof in Freiling & Yurko [21, lemma 2.3.1]. We consider the function:

$$I_R(\lambda) := \frac{1}{2\pi i}\int_{|\mu|=R} \frac{M(\mu)}{\lambda-\mu} d\mu.$$

Since $M(\lambda) = O(k^{-1})$ for $k \to \infty$, then $\lim_{R\to\infty} I_R(\lambda) = 0$. Now, we deform the contour to avoid the singularity at $\mu = \lambda$ with the little circle $\gamma_r(\lambda)$ and to avoid the cut $]0, +\infty]$. Hence,

$$\lim_{R\to 0} I_R(\lambda) = \lim_{r\to 0} \frac{1}{2\pi i}\int_{\gamma_r(\lambda)} \frac{M(\mu)}{\lambda-\mu} d\mu + \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{+\infty-i\epsilon}^{0-i\epsilon} \frac{M(\mu)}{\lambda-\mu} d\mu$$
$$+ \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{0+i\epsilon}^{+\infty+i\epsilon} \frac{M(\mu)}{\lambda-\mu} d\mu - \frac{1}{2\pi i}(2\pi i)\sum_{k=1}^{m} \operatorname{Res}\left(\frac{M(\mu)}{\lambda-\mu}\right),$$

where the last term is the sum of the residues of $\frac{M(\mu)}{\lambda-\mu}$ viewed as a function of μ. In the first term, we apply the residue theorem noticing that the little circle is run through in anti-clockwise direction; in the second term, we make the substitution $\eta = \mu + i\epsilon$; in the third term, we make the substitution $\eta = \mu - i\epsilon$, and in the last term, we replace the residue of the Weyl function with α_k (see definition 2.9):

$$0 = \frac{1}{2\pi i}(-2\pi i) \lim_{\mu\to\lambda} (\mu-\lambda)\frac{M(\mu)}{\lambda-\mu} + \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{+\infty}^{0} \frac{M(\eta-i\epsilon)}{\lambda-\eta+i\epsilon} d\eta$$
$$+ \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{0}^{+\infty} \frac{M(\eta+i\epsilon)}{\lambda-\eta-i\epsilon} d\eta - \sum_{k=1}^{m} \frac{\alpha_k}{\lambda-\lambda_k}.$$

Since $T(\eta) = \lim_{z\to 0, \operatorname{Re} z>0} \frac{1}{2\pi i}(M(\eta-iz) - M(\eta+iz))$, we can write:

$$0 = M(\lambda) + \int_{+\infty}^{0} \frac{T(\eta)}{\lambda-\eta} d\eta - \sum_{k=1}^{m} \frac{\alpha_k}{\lambda-\lambda_k},$$

which is (2.21). ∎

We can write the Weyl function $M(\lambda)$ in terms of the jump function $T(\lambda)$ and the normalizing constants α_k through the formula

$$M(\lambda) = \int_0^{\infty} \frac{T(\mu)}{\lambda-\mu} d\mu + \sum_{k=1}^{N} \frac{\alpha_k}{\lambda-\lambda_k}, \quad \lambda \in \Pi \backslash \Lambda',$$

as we can see in theorem 2.13. In order to reconstruct the Weyl function, we need to know the jump function, the eigenvalues and the normalizing constants.

Remark 2.14. *As we can see from proposition 2.10 and proposition 2.12, we can retrieve uniquely the Weyl function from the Jost function* f_h *and the eigenvalues* $\{k_j\}_{1,\ldots,N}$.

We state below a theorem from Freiling & Yurko [21, theorem 2.1.5], which proves the analyticity of $M(\lambda)$ in a certain region of the complex plane, which is related through definition 2.2 to the analyticity of the Jost function and the Jost solution (see theorem 1.9)

Theorem 2.15. *The Weyl function* $M(\lambda)$ *is analytic in* $\Pi\backslash\Lambda'$ *and continuous in* $\Pi_1\backslash\Lambda$. *The set of singularities of* $M(\lambda)$ *(as an analytic function) coincides with the set* $\Lambda_0 = \{\lambda : \lambda \geq 0\} \cup \Lambda$.

Next, we prove a uniqueness result for the Weyl function (see Freiling and Yurko [21, theorem 2.2.1]).

Theorem 2.16. (Uniqueness). *Let* V *and* $\tilde{V}$ *be in* $\mathbb{V}^1_{x_I}$ *with Weyl functions* M *and* $\tilde{M}$, *respectively. If* $M = \tilde{M}$, *then* $V = \tilde{V}$.

Proof. We closely follow the proof given in Freiling and Yurko [21]. We define the matrix $P(x,\lambda) = [P_{j,k=1,2}]$ by the formula:

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$$P(x,\lambda)\begin{bmatrix}\tilde{\varphi}(x,\lambda) & \tilde{\phi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda) & \tilde{\phi}'(x,\lambda)\end{bmatrix} = \begin{bmatrix}\varphi(x,\lambda) & \phi(x,\lambda)\\ \varphi'(x,\lambda) & \phi'(x,\lambda)\end{bmatrix}. \tag{2.22}$$

Multiplying both sides of the equation by the inverse of the matrix of the left-hand side, we get

$$P(x,\lambda) = \begin{bmatrix}\varphi(x,\lambda) & \phi(x,\lambda)\\ \varphi'(x,\lambda) & \phi'(x,\lambda)\end{bmatrix}\frac{1}{W(\tilde{\varphi},\tilde{\phi})}\begin{bmatrix}\tilde{\phi}'(x,\lambda) & -\tilde{\phi}(x,\lambda)\\ -\tilde{\varphi}'(x,\lambda) & \tilde{\varphi}(x,\lambda)\end{bmatrix}.$$

Since the Wronskian $W(\tilde{\varphi},\tilde{\phi}) = 1$ because of (2.11), we can multiply the two matrices and recover the components of the matrix $P(x,\lambda)$:

$$\begin{aligned} P_{j1}(x,\lambda) &= \varphi^{(j-1)}(x,\lambda)\tilde{\phi}'(x,\lambda) - \phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda)\\ P_{j2}(x,\lambda) &= \phi^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde{\phi}(x,\lambda). \end{aligned} \tag{2.23}$$

Solving (2.22) with respect to ϕ and φ, we get:

$$\begin{aligned} \varphi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda)\\ \phi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\phi}'(x,\lambda). \end{aligned} \tag{2.24}$$

From (2.7) and (2.3), for $|\lambda| \to \infty$, we get:

$$|P_{11}(x,\lambda) - 1| \le \frac{C}{|k|}, \qquad |P_{12}(x,\lambda)| \le \frac{C}{|k|}, \qquad |k| \to \infty. \tag{2.25}$$

From (2.23), plugging the definition of the Weyl solution $\phi(x,\lambda)$ and $\tilde{\phi}(x,\lambda)$, as in (2.9), we get:

$$\begin{aligned} P_{11} &= \varphi(x,\lambda)\tilde{\theta}'(x,\lambda) - \theta(x,\lambda)\tilde{\varphi}'(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda)\\ P_{12} &= \theta(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{\theta}(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda). \end{aligned}$$

So, if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed x, the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ. The estimates (2.25) yield $P_{11}(x,\lambda) \equiv 1$ and $P_{12}(x,\lambda) \equiv 0$. Substituting this into (2.24) we get $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and $\phi(x,\lambda) \equiv \tilde{\phi}(x,\lambda)$ for all x and λ, then $V = \tilde{V}$. ∎

Remark 2.17. *Theorem 2.16 can be found in the literature in the case of Dirichlet boundary condition under the name of Borg–Marchenko uniqueness theorem* (see [22]). *The converse of this was proved in a local version by Barry Simon (see* [23] *but also* [15,24,25]*) employing the Phragmen–Lindelöf theorem and Liouville theorem, under the assumption that if two Weyl functions asymptotically agree modulo an exponentially small function, then the two potentially agree in a certain interval.*

(c) The main equation of the inverse spectral problem

In this section, we show an alternative way (similar to Beals *et al.* [5] for the Rayleigh case) to recover the Gelfand–Levitan equation, that is an integral equation from which we can reconstruct the potential V of a Schrödinger boundary value problem. The ordinary way to obtain the Gelfand–Levitan equation is from the spectral measure. Here, we obtain it through a function ψ, that depends on the Weyl function and is discontinuous on the real line with jumps proportional to the jumps of the Weyl function. We are motivated by the fact that in the Rayleigh problem we are not able to recover the Gelfand–Levitan equation through the spectral measure as the operator is not self-adjoint, even though we will not extend the following to the Rayleigh problem.

We recall definition 2.1 and definition 2.2. We define $\phi_\pm(x,\lambda)$ as:

$$\phi_\pm(x,\lambda) = \frac{f(x,\pm k)}{f_h(\pm k)}, \qquad \mathrm{Im}\ k > 0.$$

Note that for $\lambda > 0$, we have (see also definition 2.13):

$$M^{\pm}(\lambda) = \lim_{z\to 0,\,\mathrm{Re}\,z>0} \frac{f(0,\sqrt{\lambda \pm iz})}{f_h(\sqrt{\lambda \pm iz})} = \frac{f(0,\pm k)}{f_h(\pm k)}, \qquad \lambda = k^2, \quad k>0.$$

We extend the definition of $M^{\pm}(\lambda)$ to $\lambda \in \mathbb{C}$ and note that $M^{\pm}(\lambda) = M(\lambda)$ for $\lambda \notin [0,\infty)$. In particular, it is easy to check that:

$$M^{\pm}(\lambda) = \frac{f(0,\pm k)}{f_h(\pm k)}, \qquad \mathrm{Im}\ k>0. \tag{2.26}$$

From (2.1), we can find the asymptotics, as $|k| \to \infty$, in the upper half plane $\mathrm{Im}\,k>0$ for the regular solution $\varphi(x,k)$:

$$\begin{aligned}\varphi(x,k) &= \cos kx - h\frac{\sin kx}{k} + \int_0^x \sin[k(x-t)]V(t)\varphi(t,k)dt\\ &= \left(\frac{e^{ikx}+e^{-ikx}}{2}\right) - h\left(\frac{e^{ikx}-e^{-ikx}}{2ik}\right) + \int_0^x \sin(k(x-t))\cos kt V(t)dt\\ &\quad - h\int_0^x \frac{\sin(k(x-t))}{k}V(t)\sin kt\, dt\\ &= e^{-ikx}\left(\frac{1}{2} + \int_0^x \frac{e^{2ik(x-t)}-1}{4i}V(t)dt + O\left(\frac{1}{ik}\right)\right).\end{aligned}$$

We can do the same for the Jost solution $f(x,k)$ and find its asymptotics as $|k| \to \infty$ in the physical sheet, starting from (1.11):

$$f(x,k) = e^{ikx} - \int_x^\infty \frac{e^{ik(x-t)}-e^{-ik(x-t)}}{2ik}V(t)e^{ikt}dt + O\left(\frac{1}{k^2}\right) = e^{ikx}\left(1 - \int_x^\infty \frac{V(t)}{2ik}dt + O\left(\frac{1}{k^2}\right)\right),$$

and hence

$$f(x,-k) = e^{-ikx}\left(1 + \int_x^\infty \frac{V(t)}{2ik}dt + O\left(\frac{1}{k^2}\right)\right).$$

From lemma 2.6, we have the following asymptotic expansion of $M(\lambda)$:

$$M(\lambda) = \frac{1}{ik} + \frac{1}{k^2}\left[h - \hat{V}(k)\right] + O(k^{-3}), \qquad |k| \to +\infty. \tag{2.27}$$

Since $V' \in L^1(0,\infty)$, then we can integrate by parts the Fourier transform of V and get:

$$\hat{V}(k) = -\frac{V(0)}{2ik} - \int_0^{x_I} \frac{V'(t)}{2ik}e^{2ikt}dt,$$

and (2.27) becomes:

$$M(\lambda) = \frac{1}{ik} + \frac{1}{k^2}[h - V(0)] + O(k^{-3}), \qquad |k| \to +\infty.$$

The difference between $M^+(\lambda)$ and $M^-(\lambda)$ is:

$$\frac{2}{ik} = M^+(\lambda) - M^-(\lambda) + O(k^{-3}). \tag{2.28}$$

Definition 2.18. *We define the function $\psi(x,k)$ discontinuous in the real line as:*

$$\psi(x,k) = \begin{cases} -ike^{ikx}\left(\phi_+(x,\lambda) + \frac{2i}{k}\varphi(x,k)\right) & \mathrm{Im}\,k>0\\ -ike^{ikx}\phi_-(x,\lambda) & \mathrm{Im}\,k<0,\end{cases}$$

and let $\psi_+(x,k)$ denote the restriction of $\psi(x,k)$ to the upper-half plane, and $\psi\_(x,k)$ the restriction of $\psi(x,k)$ to the lower-half plane.

One can compare definition 2.18 with Beals *et al*. [5, formula 3.8] for the Rayleigh case. We can see that the function ψ is bounded on $\mathbb{C}$. We can also write the general solution $\varphi(x,k)$ in terms of ψ_+ and $\psi\_$ as:

$$2\varphi(x,k) = e^{-ikx}\psi_+(x,k) + e^{ikx}\psi\_(x,-k).$$

Since φ is an even function of k, we also have

$$2\varphi(x,k) = e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi\_(x,k), \tag{2.29}$$

and adding these last two, we get:

$$4\varphi(x,k) = e^{ikx}(\psi_+(x,-k) - \psi\_(x,-k)) + e^{-ikx}(\psi_+(x,k) - \psi\_(x,k)).$$

From (2.9), we see that:

$$\begin{aligned} \phi_+(x,\lambda) - \phi\_(x,\lambda) &= \theta(x,k) + M^+(\lambda)\varphi(x,k) - \theta(x,k) - M^-(\lambda)\varphi(x,k) \\ &= \varphi(x,k)\left(M^+(\lambda) - M^-(\lambda)\right). \end{aligned} \tag{2.30}$$

Using (2.30) and (2.28), we can calculate:

$$\begin{aligned} \psi_+(x,k) - \psi\_(x,k) &= e^{ikx}\left(\phi_+(x,\lambda) - \phi\_(x,\lambda) - \frac{2}{ik}\varphi(x,k)\right)(-ik) \\ &= e^{ikx}\varphi(x,k)\left(M^+(\lambda) - M^-(\lambda) - \frac{2}{ik}\right)(-ik) \\ &= -ike^{ikx}\varphi(x,k)(j(k) - j(-k)), \end{aligned} \tag{2.31}$$

where $j(k)$ is of order $O(k^{-2})$, as is the coefficient of the second leading order of $M(\lambda)$, and it is defined as:

$$j(\pm k) := M^{\pm}(\lambda) \mp \frac{1}{ik}, \qquad \lambda = k^2, \quad \text{Im } k > 0,$$

and

$$j(\pm k) := M^{\pm}(\lambda) \mp \frac{1}{ik}, \qquad \lambda = k^2, \quad k > 0.$$

The difference $j(k) - j(-k)$ is:

$$j(k) - j(-k) = \begin{cases} -\dfrac{2}{ik} & \text{Im } k > 0, \\ M^+(\lambda) - M^-(\lambda) - \dfrac{2}{ik} & k > 0, \end{cases}$$

and it is of order $O(k^{-1})$ for Im $k > 0$. Now, we want to calculate the asymptotics for $\psi\_$:

$$\begin{aligned} \psi\_(x,k) &= -ike^{ikx}\phi\_(x,\lambda) = (-ik)e^{ikx}\frac{f(x,-k)}{f(0,-k)}M^-(\lambda) \\ &= (-ik)e^{ikx}\,\frac{e^{-ikx}\left(1 + \int_x^{x_I}\frac{V(t)}{2ik}dt + O(k^{-2})\right)}{1 + \int_0^{x_I}\frac{V(t)}{2ik}dt + O(k^{-2})}\,\frac{1}{-ik}\left(1 + \frac{h}{ik} - \frac{\hat{V}(-k)}{ik} + o(k^{-1})\right) \\ &= 1 + \int_x^{x_I}\frac{V(t)}{2ik}dt + \frac{h}{ik} - \int_0^{x_I}\frac{V(t)}{2ik}dt - \frac{\hat{V}(-k)}{ik} + O(k^{-2}) \\ &= 1 - \int_0^x \frac{V(t)}{2ik}dt + \frac{h}{ik} + O(k^{-2}) \end{aligned} \tag{2.32}$$

where we used that:

$$\frac{f(x,k)}{f(0,k)} = e^{ikx}\frac{\left(1 - \int_x^\infty \frac{V(t)}{2ik} + O(k^{-2})\right)}{\left(1 - \int_0^\infty \frac{V(t)}{2ik} + O(k^{-2})\right)}.$$

In the following proposition (see [2, proposition 4] for the Rayleigh case), we represent the function $\psi(x,k)$ in terms of some coefficients of the asymptotics of the Weyl function $M(\lambda)$ and its residues.

Proposition 2.19. *The function $\psi(x,k)$ satisfies:*

$$\psi(x,k) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{k' e^{ik'x}\varphi(x,k')(j(k') - j(-k'))}{k'-k} dk' + \sum_{j=1}^{N} \frac{\alpha_j}{2i(k_j+k)} e^{-ik_j x}\varphi(x,k_j) + \sum_{j=1}^{N} \frac{\alpha_j}{2i(k-k_j)} e^{ik_j x}\varphi(x,k_j)\,. \tag{2.33}$$

Moreover, the limit value $\psi_\pm(x,k) = \psi(x, k \pm i0)$ determines φ in (2.9) by:

$$2\varphi(x,k) = e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi_-(x,k)\,. \tag{2.34}$$

Proof. We consider:

$$\frac{1}{2\pi i}\int_{-R}^{R} \frac{-\psi_+(k') + \psi_-(k')}{k'-k} dk'$$

which can be written as:

$$-\frac{1}{2\pi i}\int_{-R}^{R} \frac{\psi_+(k') - 1}{k'-k} dk' + \frac{1}{2\pi i}\int_{-R}^{R} \frac{\psi_-(k') - 1}{k'-k} dk' = \lim_{\epsilon \to 0^+}\left(-\frac{1}{2\pi i}\int_{-R+i\epsilon}^{R+i\epsilon} \frac{\psi_+(k') - 1}{k'-k} dk' - \frac{1}{2\pi i}\int_{R-i\epsilon}^{-R-i\epsilon} \frac{\psi_-(k') - 1}{k'-k} dk'\right).$$

Then, we can write the integral over the interval $(-R + i\epsilon, R + i\epsilon)$ as an integral over the contour $\gamma^+(R,\epsilon)$, which consists of the arc on the upper half plane subtended by the segment $(-R + i\epsilon, R + i\epsilon)$ plus the segment itself; an integral over the arc mentioned before with opposite verse $\Gamma^+(R,\epsilon)$. We do something similar with the integral over the interval $(R - i\epsilon, -R - i\epsilon)$ that we write as an integral over the contour $\gamma^-(R,\epsilon)$, which consists of the arc on the lower half plane subtended by the segment $(R - i\epsilon, -R - i\epsilon)$ plus the segment itself; an integral over the arc mentioned above with opposite verse $\Gamma^-(R,\epsilon)$ and an integral over the positive oriented small circle γ_0 around the pole $k' = k$, that we consider lying in $\operatorname{Im} k < 0$, see figure 1.

Then,

$$-\frac{1}{2\pi i}\int_{-R+i\epsilon}^{R+i\epsilon} \frac{\psi_+(k') - 1}{k'-k} dk' - \frac{1}{2\pi i}\int_{R-i\epsilon}^{-R-i\epsilon} \frac{\psi_-(k') - 1}{k'-k} dk' - \frac{1}{2\pi i}\int_{\gamma^+(R,\epsilon)} \frac{\psi_+(k') - 1}{k'-k} dk' - \frac{1}{2\pi i}\int_{\gamma^-(R,\epsilon)} \frac{\psi_-(k') - 1}{k'-k} dk' + \frac{1}{2\pi i}\int_{\Gamma^+(R,\epsilon)} \frac{\psi_+(k') - 1}{k'-k} dk' + \frac{1}{2\pi i}\int_{\Gamma^-(R,\epsilon)} \frac{\psi_-(k') - 1}{k'-k} dk' - \frac{1}{2\pi i}\int_{\gamma_0} \frac{\psi_-(k') - 1}{k'-k} dk'$$

becomes

$$-\psi_-(x,k) + 1 - \sum_{j=1}^{N} \operatorname{Res}_{k'=k_j} \frac{\psi_+(k')}{k_j - k} + \sum_{j=1}^{N} \operatorname{Res}_{k'=-k_j} \frac{\psi_-(k')}{k_j + k} + \frac{1}{2\pi i}\int_{\Gamma^+(R,\epsilon)} \frac{\psi_+(k') - 1}{k'-k} dk' + \frac{1}{2\pi i}\int_{\Gamma^-(R,\epsilon)} \frac{\psi_-(k') - 1}{k'-k} dk'. \tag{2.35}$$

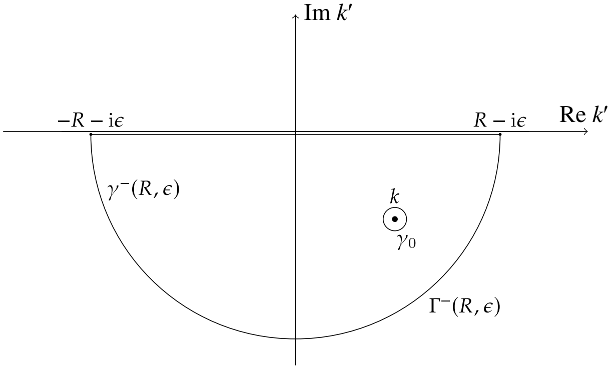

Figure 1. The closed contour $\gamma^-(R,\epsilon)$ is made by the segment from $-R - i\epsilon$ to $R - i\epsilon$ plus the arc between them in anti-clockwise way. The arc $\Gamma^-(R,\epsilon)$ is an arc from $-R - i\epsilon$ to $R - i\epsilon$ in a clockwise way. The circle γ_0 is a contour around the pole $k' = k$.

We have:

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\frac{1}{2\pi i}\int_{\Gamma^+(R,\epsilon)}\frac{\psi_+(k')-1}{k'-k}dk' + \frac{1}{2\pi i}\int_{\Gamma^-(R,\epsilon)}\frac{\psi_-(k')-1}{k'-k}dk'\right) = 0,$$

by the Jordan lemma, since $\psi_\pm - 1$ is of order $1/k'$. Thus (2.35) becomes:

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_+(k')}{k'-k}dk' + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_-(k')}{k'-k}dk' = -\psi_-(x,k)+1$$
$$+\sum_{j=1}^{N}\mathrm{Res}_{k'=-k_j}\frac{\psi_-(k')}{k_j+k} - \sum_{j=1}^{N}\mathrm{Res}_{k'=k_j}\frac{\psi_+(k')}{k_j-k}$$

so,

$$\begin{aligned}\psi(x,k)-1 &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{-\psi_+(k')+\psi_-(k')}{k'-k}dk' + \sum_{j=1}^{N}\mathrm{Res}_{k'=-k_j}\frac{\psi_-(k')}{k_j+k} - \sum_{j=1}^{N}\mathrm{Res}_{k'=k_j}\frac{\psi_+(k')}{k_j-k}\\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{-\psi_+(k')+\psi_-(k')}{k'-k}dk' + \sum_{j=1}^{N}\frac{ik_j}{k_j+k}e^{-ik_jx}\varphi(x,-k_j)\mathrm{Res}_{k'=-k_j}M^-(\lambda)\\ &\quad+\sum_{j=1}^{N}\frac{ik_j}{k_j-k}e^{ik_jx}\varphi(x,k_j)\mathrm{Res}_{k'=k_j}M^+(\lambda)\,.\end{aligned} \tag{2.36}$$

Since $\varphi(x,k)$ is even in k, we have $\varphi(x,-k)=\varphi(x,k)$. On the left-hand side, we have a meromorphic function minus its singular terms. Recalling (2.20), we have that:

$$\alpha_j = \mathrm{Res}_{\lambda'=\lambda_j} M(\lambda') = 2k_j\mathrm{Res}_{k'=k_j} M^+(\lambda) = -2k_j\mathrm{Res}_{k'=-k_j} M^-(\lambda).$$

Plugging (2.31) into (2.36), we get:

$$\psi(x,k) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k'e^{ik'x}\varphi(x,k')(j(k')-j(-k'))}{k'-k}dk'$$
$$-\sum_{j=1}^{N}\frac{i\alpha_j}{2(k_j+k)}e^{-ik_jx}\varphi(x,k_j) - \sum_{j=1}^{N}\frac{i\alpha_j}{2(k-k_j)}e^{ik_jx}\varphi(x,k_j)\,.$$

Formula (2.34) was obtained before in (2.29). ∎

In the next corollary (see [2, Corollary, page 6701]), we show the connection between the potential V and the eigenvalues k_j, the functions $j(k)$ and the normalizing constants α_j.

Corollary 2.20. *The potential V satisfies the identity:*

$$\int_0^x V(t)dt - 2h = -\frac{2i}{\pi}\int_{-\infty}^{\infty} k'\varphi(x,k')\cos(k'x)\, j(k')dk' - 2\sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\cos(k_jx) \tag{2.37}$$

and the function $\varphi(x,k)$ *satisfies*

$$\begin{aligned}2\varphi(x,k) = 2\cos(kx) - \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\varphi(x,k')j(k')\left[\frac{\sin(k'-k)x}{k'-k} + \frac{\sin(k'+k)x}{k'+k}\right]\\ - \sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\left[\frac{\sin(k_j-k)x}{k_j-k} + \frac{\sin(k_j+k)x}{k_j+k}\right].\end{aligned} \tag{2.38}$$

Proof. We start first with the proof of equation (2.37) (*Step 1*) and then we recover (2.38) (*Step 2*).

— *Step 1*. We already know the asymptotic expansion of ψ from (2.32):

$$\psi(x,k) - 1 = -\frac{1}{2ik}\int_0^x V(t)dt + \frac{h}{ik} + O(k^{-2}).$$

Multiplying $\psi - 1$ by ik and taking the limit as $k \to \infty$, we get:

$$\lim_{k\to\infty} ik(\psi(x,k) - 1) = -\frac{1}{2}\int_0^x V(t)dt + h. \tag{2.39}$$

In (2.33), we multiply by ik and take the limit as $k \to \infty$:

$$\begin{aligned}\lim_{k\to\infty} ik(\psi(x,k) - 1) &= \frac{i}{2\pi}\int_{-\infty}^{\infty} k'e^{ik'x}\varphi(x,k')(j(k') - j(-k'))dk'\\ &\quad + \frac{1}{2}\sum_{j=1}^{N}\alpha_j e^{-ik_jx}\varphi(x,k_j) + \frac{1}{2}\sum_{j=1}^{N}\alpha_j e^{ik_jx}\varphi(x,k_j)\\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty} k'e^{ik'x}\varphi(x,k')(j(k') - j(-k'))dk' + \sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\cos(k_jx)\,.\end{aligned} \tag{2.40}$$

The first term can be written as:

$$\int_{-\infty}^{\infty} k'e^{ik'x}\varphi(x,k')(j(k') - j(-k'))dk' = \int_{-\infty}^{\infty} 2k'\varphi(x,k')\cos(k_jx)\, j(k')dk'.$$

Plugging this result in (2.40) and comparing with (2.39), we get:

$$\int_0^x V(t)dt - 2h = -\frac{2i}{\pi}\int_{-\infty}^{\infty} k'\varphi(x,k')\cos(k'x)\, j(k')dk' - 2\sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\cos(k_jx).$$

— *Step 2*. We know that:

$$2\varphi(x,k) = e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi_-(x,k).$$

Both the function $\psi_+(x,-k)$ and $\psi_-(x,k)$ have poles in the lower half plane, so we can consider k in the lower half plane and use the formula (equation 2.33) for $e^{ikx}\psi_+(x,-k)$:

$$\begin{aligned}e^{ikx}\psi_+(x,-k) = e^{ikx} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')(j(k') - j(-k'))}{k'+k}dk'\\ + \sum_{j=1}^{N}\frac{\alpha_j e^{-i(k_j-k)x}}{2i(k_j-k)}\varphi(x,k_j) - \sum_{j=1}^{N}\frac{\alpha_j e^{i(k_j+k)x}}{2i(k+k_j)}\varphi(x,k_j)\,.\end{aligned}$$

The first integral can be rewritten as:

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')(j(k')-j(-k'))}{k'+k}dk' =$$
$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')j(k')}{k'+k}dk' - \frac{1}{2\pi}\int_{+\infty}^{-\infty}\frac{(-k')e^{-i(k'-k)x}\varphi(x,k')(-j(k'))}{k-k'}(-dk')$$

where the second integral after the change of variable from k' to $-k'$ becomes:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{-i(k'-k)x}\varphi(x,k')j(k')}{k'-k}dk'.$$

So, in the end we have:

$$e^{ikx}\psi_+(x,-k) = e^{ikx} - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')j(k')}{k'+k}dk' + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{-i(k'-k)x}\varphi(x,k')j(k')}{k'-k}dk'$$
$$+\sum_{j=1}^{N}\frac{\alpha_j e^{-i(k_j-k)x}}{2i(k_j-k)}\varphi(x,k_j) - \sum_{j=1}^{N}\frac{\alpha_j e^{i(k_j+k)x}}{2i(k+k_j)}\varphi(x,k_j)\,. \tag{2.41}$$

Similarly, for $e^{-ikx}\psi\_(x,k)$ we get:

$$e^{-ikx}\psi\_(x,k) = e^{-ikx} - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k'-k)x}\varphi(x,k')j(k')}{k'-k}dk' + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{-i(k'+k)x}\varphi(x,k')j(k')}{k'+k}dk'$$
$$-\sum_{j=1}^{N}\frac{\alpha_j e^{i(k_j-k)x}}{2i(k_j-k)}\varphi(x,k_j) + \sum_{j=1}^{N}\frac{\alpha_j e^{-i(k_j+k)x}}{2i(k+k_j)}\varphi(x,k_j)\,. \tag{2.42}$$

Summing (2.42) and (2.41), we get:

$$e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi\_(x,k)$$
$$= 2\cos(kx) - \frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\left[\frac{\sin(k'-k)x}{k'-k} + \frac{\sin(k'+k)x}{k'+k}\right]$$
$$-\sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\left[\frac{\sin(k_j-k)x}{k_j-k} + \frac{\sin(k_j+k)x}{k_j+k}\right].$$

∎

Equation (2.37) motivates introducing:

$$K(x,y) = \frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\cos(k'y)dk' + \sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\cos(k_j y). \tag{2.43}$$

Then equation (2.38) can be written as:

$$\varphi(x,k) = \cos(kx) - \frac{1}{2}\int_{-x}^{x}K(x,t)\cos(kt)dt = \cos(kx) - \int_{0}^{x}K(x,t)\cos(kt)dt.$$

The next proposition shows the Gelfand–Levitan equation and the algorithm one can use to recover the potential. One can compare the following proposition with Beals *et al*. [5, proposition 5].

Proposition 2.21. *The potential can be reconstructed from the Weyl function through the formula*:

$$V(x) = -2\frac{d}{dx}K(x,x), \tag{2.44}$$

where $K(x,y)$ *satisfies the Gelfand–Levitan equation:*

$$K(x,y) - g(x,y) + \frac{1}{2}\int_{-x}^{x}K(x,s)g(s,y)ds = 0, \tag{2.45}$$

with

$$g(x,y)=\begin{cases}\dfrac{i}{\pi}\displaystyle\int_{-\infty}^{+\infty}k'\cos(k'x)j(k')\cos(k'y)dk'+\sum_{j=1}^{N}\cos(k_jx)\alpha_j\cos(k_jy) & x\ge y\\ 0 & x<y\end{cases}. \tag{2.46}$$

Proof. We consider:

$$\int_{-x}^{x}K(x,y)\cos(ky)dy=\int_{-x}^{x}\left[\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\cos(k'y)dk'+\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\cos(k_jy)\right]\cos(ky)dy$$
$$=\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\int_{-x}^{x}\cos(k'y)\cos(ky)dydk'+\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\int_{-x}^{x}\cos(k_jy)\cos(ky)dy \tag{2.47}$$

and we can calculate:

$$\int_{-x}^{x}\cos(\alpha y)\cos(\beta y)dy=\frac{\sin((\alpha+\beta)x)}{\alpha+\beta}+\frac{\sin((\alpha-\beta)x)}{\alpha-\beta}.$$

Plugging this in (2.47), we get:

$$\int_{-x}^{x}K(x,y)\cos(ky)dy=\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\left[\frac{\sin((k'+k)x)}{k'+k}+\frac{\sin((k'-k)x)}{k'-k}\right]dk'$$
$$+\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\left[\frac{\sin((k_j+k)x)}{k_j+k}+\frac{\sin((k_j-k)x)}{k_j-k}\right].$$

Then, comparing with (2.38), it follows that:

$$2\varphi(x,k)-2\cos(kx)=-\int_{-x}^{x}K(x,y)\cos(ky)dy. \tag{2.48}$$

Taking into account (2.46) and (2.43), we can calculate the difference:

$$2g(x,y)-2K(x,y)=\frac{2i}{\pi}\int_{-\infty}^{+\infty}k'\cos(k'x)j(k')\cos(k'y)dk'+2\sum_{j=1}^{N}\cos(k_jx)\alpha_j\cos(k_jy)$$
$$-\frac{2i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\cos(k'y)dk'-2\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\cos(k_jy)$$
$$=2\sum_{j=1}^{N}\cos(k_jy)\alpha_j[\cos(k_jx)-\varphi(x,k_j)]+\frac{2i}{\pi}\int_{-\infty}^{+\infty}k'j(k')\cos(k'y)[\cos(k'x)-\varphi(x,k')]dk'$$

and, using formula (2.48), we get that:

$$\cos(k'x)-\varphi(x,k')=\frac{1}{2}\int_{-x}^{x}K(x,s)\cos(ks)ds.$$

Plugging this result into the previous calculations, we get:

$$2g(x,y)-2K(x,y)=\frac{i}{\pi}\int_{-\infty}^{+\infty}\int_{-x}^{x}k'j(k')\cos(k'y)K(x,s)\cos(ks)ds$$
$$+\sum_{j=1}^{N}\cos(k_jy)\alpha_j\int_{-x}^{x}K(x,s)\cos(ks)ds$$
$$=\int_{-x}^{x}K(x,s)\left[\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\cos(k'y)j(k')\cos(k's)dk'+\sum_{j=1}^{N}\alpha_j\cos(k_jy)\cos(k_js)\right]ds$$
$$=\int_{-x}^{x}K(x,s)g(s,y)ds\qquad\text{for }\ y\le s\le x.$$

Hence, the kernel $K(x,y)$ satisfies:

$$K(x,y) - g(x,y) + \frac{1}{2}\int_{-x}^{x} K(x,s)g(s,y)ds = 0.$$

Remark 2.22. (Uniqueness). Let V and $\tilde{V}$ be in $\mathbb{V}^1_{x_I}$ with Weyl functions M and $\tilde{M}$ respectively. If $M = \tilde{M}$, then $V = \tilde{V}$. Indeed, from equation (2.43), we see that:

$$K(x,x) = \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\varphi(x,k')\left(M(\lambda) - \frac{1}{ik'}\right)\cos(k'x)dk' + \sum_{j=1}^{N}\varphi(x,k_j)\cos(k_jx)2k_j\mathrm{Res}_{k'=k_j} M(\lambda)$$

and

$$\tilde{K}(x,x) = \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\tilde{\varphi}(x,k')\left(\tilde{M}(\lambda) - \frac{1}{ik'}\right)\cos(k'x)dk' + \sum_{j=1}^{N}\tilde{\varphi}(x,k_j)\cos(k_jx)2k_j\mathrm{Res}_{k'=k_j} \tilde{M}(\lambda).$$

If $M(\lambda) = \tilde{M}(\lambda)$ then $\varphi(x,k') = \tilde{\varphi}(x,k')$, which also implies $K(x,x) = \tilde{K}(x,x)$, which leads to $V(x) = \tilde{V}(x)$.

Remark 2.23. *Since* $K(x,s)$ *and* $g(s,y)$ *are even, namely* $K(x,-s) = K(x,s)$ *and* $g(-s,y) = g(s,y)$, *we can write* (2.45) *also as*:

$$K(x,y) - g(x,y) + \int_{0}^{x} K(x,s)g(s,y)ds = 0.$$

The next theorem shows for which condition the Gelfand–Levitan equation (2.45) has a unique solution (see [5, remark (ii), page 6708].

Theorem 2.24. *The Gelfand–Levitan equation (2.45) has a unique solution, for fixed* $x > 0$.

Proof. The equation (2.45) is an inhomogeneous Volterra equation, where the inhomogeneous term is $-g(x,y)$. In order to have unique solvability of (2.45), we require that the homogeneous equation:

$$K(x,y) + \int_{0}^{x} K(x,s)g(s,y)ds = 0$$

only admits the trivial solution $K(x,s) = 0$.

One can find the solution of (2.45) from the resolvent $R(s,t)$, which is obtained by iterating the kernel $g(x,y)$

$$R(s,t) = \sum_{k=0}^{\infty}(-1)^k g_{k+1}(s,t),$$

where $g_{k+1}(s,t)$ represents the $k+1$ iterate of the Volterra kernel. The solution is then:

$$K(x,y) = g(x,y) - \int_{0}^{x} R(x,t)g(t,y)dt.$$

We consider the second iterate of the kernel $g(x,y)$:

$$g_2(x,y) = \int_{0}^{x}\int_{0}^{t} g(t,s)g(s,y)dtds, \qquad 0 \le y \le s \le t \le x.$$

Since $g(t,s) = 0$ for $s > t$, we have:

$$|g_2(x,y)| = \left|\int_{0}^{x}\int_{0}^{t} g(t,s)g(s,y)dtds\right| \le \int_{0}^{x}\int_{0}^{t} \sup_{0\le s\le t}|g(t,s)| \sup_{0\le y\le s}|g(s,y)|dtds.$$

We define $d(t) := \sup_{0\le s\le t}|g(t,s)|$, so we get:

$$|g_2(x,y)| \le \int_0^x \int_0^t d(s)d(t)dtds = \frac{1}{2}\left(\int_0^x d(s)ds\right)^2.$$

Similarly,

$$|g_k(x,y)| \le \frac{1}{k!}\left(\int_0^x d(s)ds\right)^k$$

which implies that the homogeneous equation:

$$K(x,y) = -\int_0^x K(x,s)g(s,y)ds$$

admits only the trivial solution $K(x,y)=0$ as long as

$$\int_0^x \sup_{0\le s\le t} |g(t,s)|dt < \infty, \tag{2.49}$$

which holds because

$$\begin{aligned}
&\int_0^x \left| \int_{-\infty}^{\infty} k\cos(kt)\cos(ks)j(k)dk + \sum_{j=1}^{N} \cos(k_j t)\cos(k_j s)\ \alpha_j \right| dt \\
&\le \int_0^x \int_0^{\infty} |k|\,|\cos kt|\ |\cos ks|\ |j(k)-j(-k)|dkdt + c_1 x \\
&\le \int_0^x \int_0^{\infty} \frac{dkdt}{|k|^2} + c_1 x \le c_2 x .
\end{aligned} \tag{2.50}$$

∎

3. The inverse problem

In this section, we present an inverse result starting from a class $\mathbb{M}_{x_I}$ of Weyl functions to the class $\mathbb{V}^1_{x_I}$. This motivates the following definition.

Definition 3.1. (Class of Weyl function). *For fixed $h \in \mathbb{R}$, we denote by $\mathbb{M}_{x_I}$ the class of functions $M(\lambda)$ satisfying the following properties:*

(i) *$M(\lambda)$ is analytic in Π with finite number N of simple poles $\lambda_j < 0$ and residues $\alpha_j = \mathrm{Res}_{\lambda=\lambda_j} M(\lambda) > 0$.*

(ii) *$M(\lambda)$ is continuous in $\Pi_1 \setminus \{\lambda_1, \dots, \lambda_N, 0\}$ satisfying $kM(\lambda) = O(1)$ as $k \to 0$, $\mathrm{Im}\, k > 0$.*

(iii) *Let $M^{\pm}(\lambda) = \lim_{\epsilon\to 0,\, \mathrm{Re}\,\epsilon > 0} M(\lambda \pm i\epsilon)$. Then*

(iv) $M(\lambda) = \dfrac{1}{k} + \dfrac{h}{k^2} + \dfrac{V(0)}{k^2} + O(k^{-3})$, *as* $|k| \to +\infty$

(v) *The unique solution $K(x,y)$ of the Gelfand–Levitan equation (according to theorem 2.4):*

$$g(x,y) - K(x,y) + \int_0^x K(x,s)g(s,y)ds = 0$$

with

$$g(x,y) = \begin{cases} \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\cos(k'x)j(k')\cos(k'y)dk' + \sum_{j=1}^{N}\cos(k_j x)\alpha_j\cos(k_j y), & x \ge y, \\ 0, & x < y, \end{cases}$$

and $j(k) = M(\lambda) - \frac{1}{ik}$, *for any fixed* $x > 0$, *has* $K(x,x)$ *real, absolutely continuous and* $\frac{d}{dx}K(x,x) = 0$ *for* $x > x_I$ *and non-zero in a set of non-zero Lebesgue measure* $(x_I - \epsilon, x_I)$.

Remark 3.2. In condition (i) of definition 3.1, we assume that $M(\lambda)$ is analytic in Π (or equivalently for Im $k > 0$), however, the definition of $M(\lambda)$ in terms of $M^{\pm}(\lambda)$ (see (2.26)) implies that $M(\lambda)$ is analytic also in the unphysical sheet of the λ Riemann surface (and equivalently for Im $k < 0$). That is a different setting to Freiling and Yurko [21] who work only on the physical sheet (or for Im $k > 0$). Similarly, this applies to the other conditions.

In the following theorem, we characterize the class $\mathbb{V}^1_{x_I}$ by the just defined class of Weyl functions $\mathbb{M}_{x_I}$.

Theorem 3.3. *The map* $\mathcal{J}_h : \mathbb{V}^1_{x_I} \to \mathbb{M}_{x_I}$ *defined by* $\mathcal{J}_h(V) := M$ *is well-defined and bijective.*

Proof. We shall prove that, for fixed $h \in \mathbb{R}$, the map $\mathcal{J}_h$ is well-defined, that is $\mathcal{J}_h(V) = M \in \mathbb{M}_{x_I}$ for any $V \in \mathbb{V}^1_{x_I}$. The Jost function $f_h(k)$ has a finite number of zeros in $\mathbb{C}_+$ and they are all simple and pure imaginary (see [2,18]). In Sottile [2] it is proved that the Jost solution $f(x,k)$ and the Jost function $f_h(k)$ are entire in k, hence analytic for Im $k > 0$ and continuous for Im $k \geq 0$. Then, by definition of the Weyl function $M(\lambda)$, we can conclude that the Weyl function is analytic in Π, continuous in Π_1 except at the points where the denominator vanishes (see also theorem 2.15), which are the simple and pure imaginary zeros of the Jost function (see theorem 2.15 and remark 2.4. In proposition 2.12, we proved that $\alpha_j > 0$, hence condition (i) of the definition of the class of Weyl function is satisfied.

In lemma 2.11, we showed that $\frac{k}{f_h(k)} = O(1)$. Since $f(0,k) = 1 + O(1/k)$, then $kM(\lambda) = O(1)$, which implies condition (ii).

Condition (iii) is proved by proposition 2.10 since for $\lambda > 0$ we get $T(\lambda) = \frac{\sqrt{\lambda}}{\pi |f_h(k)|^2} > 0$. By lemma 2.6, we can see that condition (iv) is satisfied.

From proposition 2.21 and theorem 2.24, we see that condition (v) is satisfied, hence $M \in \mathbb{M}_{x_I}$ and $\mathcal{J}_h$ is well-defined.

The injectivity of the map $\mathcal{J}_h$ is given by theorem 2.16.

To prove subjectivity, we fix $M(\lambda) \in \mathbb{M}_{x_I}$ and we want to prove that there exists a $V \in \mathbb{V}^1_{x_I}$ such that $\mathcal{J}_h(V) = M(\lambda)$. Conditions (i–iv) allow us to define a function $g(x,y)1$ as in (2.46) and $K(x,y)$ which satisfies the Gelfand–Levitan equation (see proposition 2.21). From $K(x,y)$, solution of (2.45), we can construct (as in ((2.48)):

$$\varphi(x,k) = \cos kx - \int_0^x K(x,y) \cos(ky) dy,$$

that is a solution to the boundary value problem (1.7), (1.8) with $V(x) = -2\frac{d}{dx}K(x,x)$, and $h = K(0,0)$ given.

From theorem 2.24 and condition (v), we know that the Gelfand–Levitan equation (2.45) has a unique solution $K(x,y)$, such that $V = -2\frac{d}{dx}K(x,x)$ is in the class $\mathbb{V}^1_{x_I}$. ∎

The reader can compare definition 3.1 and theorem 3.3 with the definition of the class **W** and theorem 2.2.5 in Freiling and Yurko [21], which are obtained for a different class of potentials and through a different Gelfand–Levitan equation.

Algorithm 3.4. *Starting from a set of eigenvalues and resonances* $\{k_j\}_1^\infty$ *we can retrieve the potential* $V_\omega(x)$ *using the following algorithm:*

- *Construct the Jost function from*

$$f_h(k) = f_h(0)e^{ik} \lim_{R\to\infty} \prod_{|k_n|\le R} \left(1 - \frac{k}{k_n}\right),$$

where $f_h(0)$ *is determined so that* $f_h(k) = ik + O(1)$ *as* $k \to \infty$.

- *From* $\{k_j\}_1^\infty$ *and* $f_h(k)$ *we construct the jump function* $T(\lambda)$ *and the normalizing constant* α_k *through formulas* (2.14) *and* (2.18)*:*

$$T(\lambda) = \frac{k}{\pi |f_h(k)|^2},$$

$$\alpha_j = 4k_j^2 \left[\frac{-i}{f_h(-k_j)\dot{f}_h(k_j)}\right].$$

- *Use the spectral data* $\left(T(\lambda), \{\alpha_j, \lambda_j\}_{j=1,\dots,N}\right)$ *to construct the Weyl function via formula* (2.21)

$$M(\lambda) = \int_0^\infty \frac{T(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^N \frac{\alpha_k}{\lambda - \lambda_k}, \quad \lambda \in \Pi \backslash \Lambda'.$$

- *Then construct* $g(x, y)$ *in* (2.46) *as in*

$$g(x,y) = \begin{cases} \frac{i}{\pi}\int_{-\infty}^{+\infty} k' \cos(k'x) j(k') \cos(k'y) dk' + \sum_{j=1}^N \cos(k_j x)\alpha_j \cos(k_j y), & x \ge y \\ 0, & x < y \end{cases},$$

where $j(k) := M(\lambda) - \frac{1}{ik}$.

- *Solve the Gelfand–Levitan equation* (2.45) *with respect to* $K(x, y)$,

$$K(x,y) - g(x,y) + \frac{1}{2}\int_{-x}^{x} K(x,s) g(s,y) ds = 0.$$

- *Obtain the potential from* (2.44)*:*

$$V_\omega(x) = -2\frac{d}{dx} K(x,x).$$

- *Obtain the shear modulus from*

$$\hat{\mu}(x) = \frac{\hat{\mu}_I\left(\omega_1^2 - \omega_2^2\right)}{\omega_1^2 - \omega_2^2 - \hat{\mu}_I\left(V_{\omega_1}(x) - V_{\omega_2}(x)\right)}. \tag{3.1}$$

Remark 3.5. *The formula (recovery of Lamé parameter) for the reconstruction of the Lamé parameter* $\hat{\mu}$ *is obtained in* [2, *theorem 4.9]* *utilizing* (1.6).

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