REMARKS ON BIFURCATION IN ELLIPTIC BOUNDARY VALUE PROBLEMS

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In this note we present some observations of bifurcation in nonlinear elliptic boundary value problem $-\Delta u = f(\lambda, u)$, in Ω , $u = 0$ on $\partial\Omega$. In particular, we are interested in the effect of concave and convex combination of Ambrosetti, Brezis and Cerami type and get new class of concave and convex nonlinearity.

1. Introduction

Consider the nonlinear elliptic boundary value problem

$$\Delta u = f(\lambda, u), \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1)$$

where $\lambda > 0$ is a bifurcation parameter. A fundamental question here is: how many solutions does it have and how does the solution(s), if it exists, depend on the parameter? As it has many impacts on other subjects and has attracted great attention for several decades. In general, the "uniqueness" of the solution is rare and the "multiplicity" of the solution is generic. In this note, we shall show some observations concerning the multiplicity. Many of our examples are in a character of numerical simulations.

2. Positive solutions

The positive solution is unique for the problem with sublinearity in the sense of that $f(\lambda, u)/u$ is decreasing on $(0, \infty)$ [2]. For the superlinear problem, the uniqueness is known only for special class of nonlinearity on balls or the whole space [7]. But if the dimension is one, then the uniqueness remains valid, if $f(\lambda, u)/u$ is increasing on $(0, \infty)$ [4]. Now we consider perturbed

linear problem and let $f = \lambda u(1 + \varepsilon(u))$, where $\varepsilon(u)$ is a perturbation. If we have a small negative (positive) $\varepsilon(u)$ then we actually push the straight line up (down), and therefore produce example with more solutions (see the figure below). To be precise, let $\varphi(u, a, b) = 1 - \cos(2\pi(u - a)/(b - a))$, $u \in (a, b)$, and $= 0$, otherwise.

Example Consider the 1D problem on $(-1, 1)$ with perturbed non-linearity $f = \lambda u(1 + \frac{1}{4}\varphi(u, \frac{\pi}{4}, \frac{\pi}{2}) - \frac{3}{8}\varphi(u, \frac{3\pi}{4}, \pi) + \frac{1}{8}\varphi(u, \frac{5\pi}{4}, \frac{3\pi}{2}))$. Then we can compute the graph of $f/(\lambda u)$ (left figure) as well as the bifurcation diagram (right figure)

Another example of multiplicity is the concave and convex combination of Ambrosetti, Brezis and Cerami type[1], $f(\lambda, u) = u^p + \lambda u^q$, $0 < q < 1 < p$, where one can have a) exact two solutions, b) a unique solution or c) no solution at all. If we perturb the above nonlinearity in the following way, $f(\lambda, u) = \lambda(\alpha u^q + u^p - \varepsilon u^{p_1} + \delta u^{p_2})$, where $0 < q < 1 < p < p_1 < p_2$, and parameters $\varepsilon, \delta \geq 0$ small, then we have the bifurcation below of four solutions. (In the left figure, $f = \lambda(0.11u^{0.1} + 2.3u^{1.3} - 0.3u^2 + 0.01512u^{2.6})$)

A new class of concave and convex combination is the following quasilinear problem [2]:

$$-(\Delta + \varepsilon \Delta_p)u = \lambda u^q, \quad (2)$$

where $p > 1, \varepsilon > 0$ and q is in between of 2 and p , $\Delta_p u = \operatorname{div}\{| \nabla u |^{p-2} \nabla u\}$ is the p -Laplacian.

Theorem 2.1. Suppose that $p > 1, \varepsilon > 0$, either $2 < q < p$ or $1 < q < 2$, and $\Omega = (-L, L)$, then there is $\lambda_0 > 0$ such that the above quasilinear equation (2) has two positive solutions, if $\lambda > \lambda_0$ and has no positive solution if λ stays below λ_0 . (figure in the left below)

The second new class of concave and convex combination is $f = \lambda(u - \alpha u^p + \beta u^q)$, $\alpha, \beta > 0, q > p > 1$, where function $g = u - \alpha u^p$ is concave, while $h = \beta u^q$ is convex. In case f is positive on $(0, \infty)$, which occurs precisely as $(q-1)^{q-1}\beta^{p-1} > (p-1)^{p-1}\alpha^{q-1}(q-p)^{q-p}$, then we shall have the bifurcation diagram (figure in the right) as above.

Theorem 2.2. *Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary, then under the above assumptions, there is $\Lambda(> \lambda_1)$ such that the problem (1) has two, one or no positive solutions, if $\lambda \in (\lambda_1, \Lambda)$ respectively $\lambda \in (0, \lambda_1]$ or $\lambda > \Lambda$.*

3. Sign-changing solutions

In this part, we restrict ourself to one dimensional problem

$$-\Delta u = f(\lambda, u), \quad -1 < x < 1, \quad u(-1) = u(1) = 0. \quad (3)$$

If f is odd in u , then the nodal solutions are just periodic extensions of the scaled positive solution. In the sequent we will consider the case $f_+(u) \neq$

$f_-(u)$, where $f_+(u) = f(u), u \geq 0, f_+(u) = 0, u < 0, f_-(u) = -f(-u), u \leq 0, f_-(u) = 0, u > 0$. Let $F_{\pm}(u) = \int_0^u f_{\pm}(s) ds$, and $u(x)$ be nodal solution of (3) with m upper waves and n lower waves, $|m-n| \leq 1, 1u_0 = \max\{u(x)\} > 0, v_0 = \max\{-u(x)\} > 0$, then by the time-mapping analysis [4, 5, 6], we derive that the following equations hold: $F_+(\lambda, u_0) = F_-(\lambda, v_0)$, and

$$m \int_0^1 \frac{u_0 d\theta}{\sqrt{F_+(\lambda, u_0) - F_+(\lambda, \theta u_0)}} + n \int_0^1 \frac{v_0 d\theta}{\sqrt{F_-(\lambda, v_0) - F_-(\lambda, \theta v_0)}} = \sqrt{2}$$

It is apparently much harder to analysis the above system of nonlinear equations than the study of time mapping of positive solutions. To illustrate the richness of system above, we pick up function $f = \lambda(6u_+^5 + 1.5u_+^{0.5} - 2.1u_-^{1.1} + 156u_-^{0.2})$, as an example, $u_+ = \max\{u, 0\}, u_- = \max\{-u, 0\}$. We can study the bifurcations of λ via u_0 (figure in the left below) as well as the life spans of upper, and lower waves (figure in the right) with help of MatLab, and get the following bifurcation results for the solutions with 2 interior nodes.

For solutions with four nodes, we get the result shown by the figure in the right. We also observe that the problem is invariant under reflection $x \rightarrow -x$, hence $u(-x)$ solves the same equation (figure in the left). As a consequence, we obtain eight solutions having four nodes for certain interval of bifurcation parameter λ . (figure in the right)

References

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