Global Stability Analysis on the Dynamics of an SIQ Model with Nonlinear Incidence Rate

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Abstract. An SIQ epidemic model with isolation and nonlinear incidence rate is studied. We have obtained a threshold value R and shown that there is only a disease free equilibrium point when $R < 1$, and there is also an endemic equilibrium point if $R > 1$. With the help of Liapunov function, we have shown that disease free- and endemic equilibrium point is globally stable.

Keywords: SIQ model, Isolation, nonlinear infectious rate, threshold, equilibrium point, global stability.

1 Introduction

In this paper, we shall as well exploit the Lyapunov direct method to study the global stability analysis on the dynamics of an SIQ model with nonlinear incidence rate. Assuming that the population has a very short immunity period, which is negligible, and whence there is a class of recovered and the models being studied here are of SIQ type. Assume that the total population is divided into: $S = S(t)$ susceptible, $I(t)$ infected, $Q(t)$ isolated and $R(t)$ recovered classes at time t . Let A be the constant migration rate of the population, and d be natural death rate of each kind, α_1 and α_2 be the infected death rate of infectious and isolated classes respectively, and δ be the transition rate from infected to isolated classes, r, ε be immunity's lose rate of infected respectively isolated classes. We assume that the constants d, δ, A are positive, and $r, \varepsilon, \delta, \alpha_1, \alpha_2$ are nonnegative. Thus we consider the

$$\begin{cases} S'(t) = A - \frac{\beta IS}{1 + mI} - dS + rI + \varepsilon Q \\ I'(t) = \frac{\beta IS}{1 + mI} - (r + \delta + d + \alpha_1)I \\ Q'(t) = \delta I - (\varepsilon + d + \alpha_2)Q \end{cases} \quad (1)$$

we see that N satisfies: $N'(t) = A - dN - \alpha_1 I - \alpha_2 Q \leq A - dN$, thus

$$N(t) \leq N_0 e^{-dt} + \frac{A}{d}(1 - e^{-dt}) \quad (2)$$

It follows from the above estimates that the solutions of initial value problem for system

(1) exist on $[0, +\infty)$, $D = \{(S, I, Q) \in R_+^3 : S, I, Q \geq 0, S + I + Q \leq A/d\}$ is invariant subset of (1).

2 Local Stability of Equilibrium

To find the equilibrium of system (1), we get the equilibrium $P_0(A/d, 0, 0)$ and $P_1(S_1, I_1, Q_1)$, let's define the threshold number for (1)

$R_1 = \frac{d(r + \delta + d + \alpha_1)}{\beta A}$, Then we have that P_1 is in R_+^3 if and only if $R_1 < 1$.

Here $S_1 = \frac{(r + \delta + d + \alpha_1)}{\beta}(1 + mI_1)$,

$$Q_1 = \frac{\delta I_1}{\varepsilon + d + \alpha_2}, \quad I_1 = \frac{\beta A - d(r + \delta + d + \alpha_1)}{md(r + d + \delta + \alpha_1) + \beta(d + \alpha_1 + \frac{\delta(d + \alpha_2)}{\varepsilon + d + \alpha_2})}$$

Theorem 1. For the system (1), the infection-free equilibrium point P_0 is locally asymptotically stable if $R_1 > 1$, If $R_1 < 1$, then the endemic equilibrium point P_1 is locally asymptotically stable and P_0 becomes unstable.

Proof. First we consider the system (1) and the infectious free equilibrium P_0 , The Jacobian matrix $J(P_0)$, Which has eigenvalues:

$$\lambda_1 = -(d + \varepsilon) < 0, \quad \lambda_2 = \frac{\beta A}{d} - (r + \delta + d + \alpha_1), \quad \lambda_3 = -(\varepsilon + d + \alpha_2) < 0,$$

When $R_1 < 1$, then λ_2 is also negative, and hence P_0 is locally stable.

In the P_1 , $\beta S_2 = (r + d + \delta + \alpha_1)(1 + mI_2)$. Let $d_i = d + \alpha_i, i = 1, 2$

According to $J(P_1)$, the associated characteristic equation is $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$,

$$b_1 = \beta I_2 + d(1 + mI_2) + mI_2(r + \delta + d_1) + (\varepsilon + d_2)(1 + mI_2) > 0$$

$$b_2 = (\beta I_2 + d + dmI_2)mI_2(r + \delta + d_1) + (\beta I_2 + d + dmI_2)(\varepsilon + d_2)(1 + mI_2) + mI_2(r + \delta + d_1)(\varepsilon + d_2)(1 + mI_2) + \beta I_2(\delta + d_1 - rmI_2) > 0$$

$$b_3 = mI_2(\varepsilon + d_2)(1 + mI_2)(r + \delta + d_1)(\beta I_2 + d + dmI_2) + \beta I_2((\varepsilon + d_2)(1 + mI_2)(\delta + d_1 - rmI_2) - \delta\varepsilon(1 + mI_2)^2) > 0$$

To verify the Hurwitz condition $b_1b_2 - b_3 > 0$, we first rewrite:

$$b_1 = c_{10} + c_{11}mI_2,$$

$$b_2 = c_{20} + c_{21}mI_2 + c_{22}(mI_2)^2, \quad b_3 = c_{30} + c_{31}mI_2 + c_{32}(mI_2)^2 + c_{33}(mI_2)^3,$$

Where $c_{10} = d + \varepsilon + d_2 + \beta I_2$, $c_{11} = d + r + \varepsilon + d_2 + \delta + d_1$,

$$c_{20} = d(\varepsilon + d_2) + (\varepsilon + d_2 + \delta + d_1)\beta I_2,$$

$$c_{21} = rd + (r + \delta + d_1)(\varepsilon + d_2) + (d + \beta I_2)(\varepsilon + d_2 + \delta + d_1),$$

$$c_{22} = (r + \delta + d_1)(\varepsilon + d_2) + d(r + \varepsilon + d_2 + \delta + d_1),$$

$$c_{30} = (\varepsilon d_1 + \delta d_2 + d_1 d_2)\beta I_2,$$

$$c_{31} = d(r + \delta + d_1)(\varepsilon + d_2) + 2(\varepsilon d_1 + \delta d_2 + d_1 d_2)\beta I_2,$$

$$c_{32} = 2d(r + \delta + d_1)(\varepsilon + d_2) + (\varepsilon d_1 + \delta d_2 + d_1 d_2)\beta I_2,$$

$$c_{33} = d(r + \delta + d_1)(\varepsilon + d_2).$$

Then get

$$b_1b_2 - b_3 = c_{10}c_{20} - c_{30} + (c_{10}c_{22} + c_{11}c_{20} - c_{31})mI_2 +$$

$$(c_{10}c_{22} + c_{11}c_{21} - c_{32})(mI_2)^2 + (c_{11}c_{22} - c_{33})(mI_2)^3$$

$$C_0 =: c_{10}c_{20} - c_{30} > 0, C_1 =: c_{10}c_{22} + c_{11}c_{20} - c_{31} > 0,$$

$C_2 =: c_{10}c_{22} + c_{11}c_{21} - c_{32} > 0$, $C_3 =: c_{11}c_{22} - c_{33} > 0$, So P_1 is locally asymptotically stable.

3 Global Stability of Equilibrium

Theorem 2. For the system (1), If $R_1 > 1$, then the disease free equilibrium P_0 of (2) is globally asymptotically stable; when $R_1 < 1$, then P_0 is unstable, and the endemic equilibrium point P_1 is globally asymptotically stable.

Proof. we can easily proof the disease free equilibrium P_0 of (1) is globally asymptotically stable same as theorem. Following discuss asymptotic stability of P_1 , we again change variable $N = S + I + Q$

$$\begin{cases} N'(t) = A - dN - \alpha_1 I - \alpha_2 Q \\ I'(t) = \frac{I}{1+mI} [\beta(N - I - Q) - (r + \delta + d + \alpha_1)(1 + mI)] \\ Q'(t) = \delta I - (\varepsilon + d + \alpha_2)Q \end{cases}$$

$x = N - N_1, y = I - I_1, z = Q - Q_1$, then we deduce

$$\begin{cases} x'(t) = -dx - \alpha_1 y - \alpha_2 z \\ y'(t) = \frac{y + I_2}{1 + m(y + I_2)} (\beta x - (\beta + m(r + d + \delta + \alpha_2))y - \beta z) \\ z'(t) = \delta y - (\varepsilon + d + \alpha_2)z \end{cases} \quad (3)$$

We have used the relation $(r + \delta + d + \alpha_1)(1 + mI_2) = \beta S_2 = \beta(N_2 - I_2 - Q_2)$

Choose the Liapunov function

$$V = \frac{\omega_1 x^2}{2} + \omega_2 (y - I_2 \ln(1 + \frac{y}{I_2}) + \frac{my^2}{2}) + \frac{\omega_3 z^2}{2} + \frac{(x - z)^2}{2}$$

$$\left. \frac{dV}{dt} \right|_{(3)} = \omega_1 x(-dx - \alpha_1 y - \alpha_2 z) + \omega_2 y(\beta x - (\beta + m(r + d + \delta + \alpha_2))y - \beta z)$$

$$+ \omega_3 z(\delta y - (\varepsilon + d + \alpha_2)z) + (x - z)(-dx - (\delta + \alpha_1)y + (\varepsilon + d)z)$$

$$= -d(1 + \omega_1)x^2 + (\omega_2\beta - \omega_1\alpha_1 - \alpha_1 - \delta)xy + (\varepsilon + 2d - \alpha_2\omega_1)xz - \omega_2(\beta + m(r + d + \delta + \alpha_2))y^2$$

$$+ (\alpha_1 + \delta + \omega_3\delta - \omega_2\beta)yz - (\varepsilon + d + (\varepsilon + d + \alpha_2)\omega_3)z^2$$

If we choose $\omega_1, \omega_2, \omega_3 > 0$ $\omega_1 = \frac{(\varepsilon + 2d)}{\alpha_2}, \omega_2 = \frac{(\omega_1\alpha_1 + \alpha_1 + \delta)}{\beta}, \omega_3 = \frac{\omega_1\alpha_1}{\delta}$,

$$\left. \frac{dV}{dt} \right|_{(3)} = -d(1 + \omega_1)x^2 - \omega_2(\beta + m(r + d + \delta + \alpha_2))y^2 - (\varepsilon + d + (\varepsilon + d + \alpha_2)\omega_3)z^2$$

Which is clearly negative definite. Hence, $P_1(S_1, I_1, Q_1)$ is globally asymptotically stable.

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