A model for graded materials with application to cracks

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Received 23 October 2002; accepted 1 October 2003

Abstract. Stress intensity factors are calculated for long plane cracks with one tip interacting with a region of graded material characteristics. The material outside the region is considered to be homogeneous. The analysis is based on assumed small differences in stiffness in the entire body. The linear extent of the body is assumed to be large compared with that of the graded region. The crack tip, including the graded region, is assumed embedded in a square-root singular stress field. The stress intensity factor is given by a singular integral. Solutions are presented for rectangular regions with elastic gradient parallel to the crack plane. The limiting case of infinite strip is solved analytically, leading to a very simple expression. Further, a fundamental case is considered, allowing the solution for arbitrary variation of the material properties to be represented by Fourier's series expansion. The solution is compared with numerical results for finite changes of modulus of elasticity and is shown to have a surprisingly large range of validity. If an error of 5% is tolerated, modulus of elasticity may drop by around 40% or increase with around 60%.

Key words: Asymptotic analysis, elastic material, fracture toughness, inhomogeneous material, stress intensity factor.

1. Introduction

Functional layers and protective coatings are increasingly used to improve mechanical, thermal and chemical performance of tools and devices. As an example, fatigue cracks are attracted by a bi-material interface to a weaker or softer material and they are retarded by a bi-material interface to a stiffer or harder material (Suresh et al., 1992). These phenomena might be used to attract and trap fatigue cracks in a layer of soft or weak material. Another example is the improved wear resistance that successfully has been obtained by applying hard coating surfaces to tools. Here the effect of material changes on the crack tip driving force is of interest.

The problem for a crack advancing in a bi-material solid was firstly treated by Zak and Williams (1963) and consequently by, e.g., Atkinson (1975), Lu and Erdogan (1983) He and Hutchinson (1989) and Romeo and Ballarini (1995). A series expansion for the stresses around the crack tip was deduced. One result of the analysis was that the strength of the singular stress field surrounding the crack tip is generally weaker or stronger than the well-known square root singular field. This causes analytical difficulties whereas it leads to either infinite or vanishing stress intensity factors for a crack that approaches the interface. The prediction is that the crack either becomes unconditionally unstable or impossible to drive through the interface. One remedy to that is to consider the linear extent of the crack tip process region (Romeo and Ballarini, 1995). However for brittle material the predictions still become unrealistic (Wäppling et al., 1998).

Modern processes for diffusive or evaporative applications of thin layers to surfaces and thin internal layers in structures have provided possibilities to manufacture layers of size of

few atomic distances. The technique introduces new possibilities to increase the lifetime of components e.g. in conjunction with manufacturing of electronic devices. During their production, as they are exposed to elevated temperatures, considerable diffusion occurs across the material boundaries, naturally creating material properties gradients. Thus the bonding between dissimilar materials defines interfaces with spatially varying mechanical characteristics. This might question the assumption regarding a distinct bi-material interface. However, the diffusion may be taken advantage of, since graded materials with tailored material property distribution may be developed. The technique for this is known but has not been used to optimise the fracture mechanical properties of the composite.

The fracture mechanical aspects of functionally graded materials have been dealt with in a number of works. An exponential spatial variation of the elastic modulus has been assumed in the analysis by Atkinson and List (1978), and Delale and Erdogan (1983). Further it has been shown (Erdogan et al., 1991; Erdogan, 1995) that assuming a smooth material property distribution at the interface between a material with constant properties and a functionally graded material, the anomalous phenomenon of either infinite or vanishing stress intensity factors is eliminated.

In the present paper a graded material with varying elastic properties is studied. A general model, where the non-homogeneity is treated as a perturbation of some constant characteristics is described Section 2. Section 3 specialises the analysis to plane cases and shows the way for solving such problems. A crack approaching, penetrating and passing a region of graded material is considered in Section 4. A general solution for the stress intensity factor variation in case of arbitrary shaped region is given in terms of area integral. Special interest is devoted to a strip shaped layer with a material gradient in the direction perpendicular to the extent of the strip, and a crack perpendicular to the strip. The solution for a layer is shown to be a linear combination of the gradient function and its Hilbert transform. The results of the analyses are presented in Section 5 for a particular choice of the gradient function. The analytical solution for this case is compared to numerical results from FE calculations (Jivkov and Ståhle, 2000). A very good agreement is shown for large range of gradient magnitudes. Finally, it is demonstrated how this special choice of the gradient function allows the result for an arbitrary function to be written in the form of a trigonometric series expansion.

2. General model

Consider an infinite isotropic linear elastic body of material given by the Lamé constants μ^0 and λ^0 . An inclusion, identified as an arbitrarily shaped region of a different material, is introduced into the otherwise homogeneous body. The domain occupied by the inclusion is denoted by G and a coordinate system (x_1, x_2, x_3) is introduced. The material of the inclusion is given by the Lamé parameters $\mu = \mu(x_1, x_2, x_3)$ and $\lambda = \lambda(x_1, x_2, x_3)$, that are bounded functions in G and assume the values μ^0 and λ^0 respectively on the boundary of G . For simplicity, stress boundary conditions are considered everywhere.

It is assumed that the Lamé parameters may be expanded as follows:

$$\mu = \mu^0 + \Delta\mu^1 + O(\Delta)^2, \quad (1a)$$

$$\lambda = \lambda^0 + \Delta\lambda^1 + O(\Delta)^2, \quad (1b)$$

as $\Delta \rightarrow +0$, where Δ is a positive non-dimensional constant.

It is further assumed that stresses and strains can be expanded in the same way. The following is suggested

$$\sigma_{ij} = \sigma_{ij}^0 + \Delta\sigma_{ij}^1 + O(\Delta)^2, \quad (2a)$$

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \Delta\varepsilon_{ij}^1 + O(\Delta)^2, \quad (2b)$$

as $\Delta \rightarrow +0$, where the Latin indices assume values 1, 2 and 3.

Stresses and strains are related through Hooke's law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}. \quad (3)$$

Insertion of (2a) and (2b) into (3) gives

$$\sigma_{ij}^0 = 2\mu^0\varepsilon_{ij}^0 + \lambda^0\varepsilon_{kk}^0\delta_{ij}. \quad (4)$$

and

$$\sigma_{ij}^1 = \sigma_{ij}^* + 2\mu^1\varepsilon_{ij}^0 + \lambda^1\varepsilon_{kk}^0\delta_{ij}, \quad (5)$$

where a pseudo-stress σ_{ij}^* collects the part dependent of first order perturbation of strain terms.

Thus, σ_{ij}^* is given by

$$\sigma_{ij}^* = 2\mu^0\varepsilon_{ij}^1 + \lambda^0\varepsilon_{kk}^1\delta_{ij}. \quad (6)$$

Equilibrium requires that

$$\sigma_{ij,j}^1 = 0, \quad (7)$$

and therefore it is necessary that

$$\sigma_{ij,j}^* = -(2\mu^1\varepsilon_{ij}^0 + \lambda^1\varepsilon_{kk}^0\delta_{ij})_{,j}. \quad (8)$$

The solution (2a) to the perturbed problem is now given by

$$\sigma_{ij} = \sigma_{ij}^0 + \Delta(\sigma_{ij}^* + 2\mu^1\varepsilon_{ij}^0 + \lambda^1\varepsilon_{kk}^0\delta_{ij}) + O(\Delta)^2, \quad (9)$$

where σ_{ij}^* fulfil (8). The inclusion is assumed fully embedded in the body. Thus, since μ^1 and λ^1 are assumed to vanish on all boundaries, it is also required that any traction caused by σ_{ij}^* vanishes on all boundaries. This provides boundary conditions to complete Equation (8). Once the zeroth order solution is determined, the remaining problem is to find σ_{ij}^* via (8), where the right hand side is known.

3. Plane problems

The body is assumed to be in a state of either plane strain or generalised plane stress. The stresses σ_{ij}^0 in a corresponding homogeneous body may be represented by two complex functions $\psi(z)$ and $\phi(z)$, where $z = x_1 + ix_2$ (see, e.g. Muschelisvili, 1963). The analytic functions $\psi(z)$ and $\phi(z)$ are related to the stress components through

$$\sigma_{11}^0 + \sigma_{22}^0 = 2 \left[\psi(z) + \overline{\psi(z)} \right], \quad \sigma_{22}^0 - \sigma_{11}^0 - 2i\sigma_{12}^0 = 4\overline{\phi(z)} + 2(z - \bar{z})\overline{\psi'(z)}. \quad (10)$$

For plane strain and generalised plane stress the inverse of (3) may be written as

$$\varepsilon_{\alpha\beta} = \frac{1}{2\mu} \left[\sigma_{\alpha\beta} + \frac{\kappa - 3}{4} \sigma_{\gamma\gamma} \delta_{\alpha\beta} \right], \quad (11)$$

where Greek indices assume values 1 and 2. The parameter κ is defined by

$$\kappa = \begin{cases} \frac{3\mu + \lambda}{\mu + \lambda} & \text{plane strain,} \\ \frac{6\mu + 5\lambda}{2\mu + 3\lambda} & \text{plane stress.} \end{cases} \quad (12)$$

The expansions (1a) and (1b) imply that κ may be expanded as follows:

$$\kappa = \kappa^0 + \Delta \kappa^1 + O(\Delta)^2, \quad (13)$$

as $\Delta \rightarrow +0$. It is easy to show that the connection between the first variations of λ , μ and κ reads

$$\lambda^1 = \begin{cases} \frac{3 - \kappa^0}{\kappa^0 - 1} \mu^1 - \frac{2\mu^0}{(\kappa^0 - 1)^2} \kappa^1 & \text{plane strain,} \\ \frac{2(3 - \kappa^0)}{3\kappa^0 - 5} \mu^1 - \frac{8\mu^0}{(3\kappa^0 - 5)^2} \kappa^1 & \text{plane stress} \end{cases} \quad (14)$$

The right-hand side of (8) may be viewed as an analogue to a body force. Therefore, to solve (8) it becomes convenient to rewrite the equations of equilibrium as follows

$$\sigma_{1\alpha,\alpha}^* + i\sigma_{2\beta,\beta}^* + F = 0, \quad (15a)$$

or in complex form

$$\frac{\partial}{\partial \bar{z}} (\sigma_{11}^* + \sigma_{22}^*) - \frac{\partial}{\partial z} (\sigma_{22}^* - \sigma_{11}^* - 2i\sigma_{12}^*) + F = 0. \quad (15b)$$

The quantity F , defined in the last equation, may be derived in terms of zeroth order strains or stresses using (3), (8), (11) and (14) as follows

$$F = \frac{\partial}{\partial \bar{z}} \left[4 \left(\frac{\mu^1}{\kappa^0 - 1} - \frac{\mu^0 \kappa^1}{(\kappa^0 - 1)^2} \right) (\varepsilon_{11}^0 + \varepsilon_{22}^0) \right] - \frac{\partial}{\partial z} [2\mu^1 (\varepsilon_{22}^0 - \varepsilon_{11}^0 - 2i\varepsilon_{12}^0)] = \quad (16)$$

$$\frac{\partial}{\partial \bar{z}} \left[\left(\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0 - 1} \right) (\sigma_{11}^0 + \sigma_{22}^0) \right] - \frac{\partial}{\partial z} \left[\frac{\mu^1}{\mu^0} (\sigma_{22}^0 - \sigma_{11}^0 - 2i\sigma_{12}^0) \right]$$

After inserting (10) into the last equation, the following remains

$$F = 4 \left\{ \frac{\partial}{\partial \bar{z}} \left[\left(\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0 - 1} \right) \frac{\psi + \bar{\psi}}{2} \right] - \frac{\partial}{\partial z} \left[\frac{\mu^1}{\mu^0} \left(\bar{\phi} + \frac{z - \bar{z}}{2} \bar{\psi}' \right) \right] \right\}. \quad (17)$$

The real part of the complex quantity F represents a force in the direction of x_1 and its imaginary part represents a force in the direction of x_2 applied at the position z . The stresses σ_{ij}^* due to these body forces solve (8) and are obtained with any of a large variety of standard methods. Given the stress σ_{ij}^* , the total stress σ_{ij} is determined by equations (2a) and (5), where the right hand side of (5) now contains only known quantities.

Figure 1. Schematic of a crack approaching an arbitrarily shaped inclusion.

4. Analysis of a crack

Consider a large plane body containing a crack. The material is assumed homogeneous in the body, apart from an inclusion G (see Figure 1). The crack occupies the region $x_1 < 0, x_2 = 0$. It is assumed in addition, that the crack length is much larger than the characteristic lengths of the inclusion, as well as the distance between the crack tip at $x_1 = x_2 = 0$ and the inclusion.

The crack surfaces are presumed traction free, i.e.,

$$\sigma_{22} = \sigma_{12} = 0 \quad \text{for } x_1 < 0, x_2 = 0. \quad (18)$$

At large distances the stresses are given by the following:

$$\sigma_{ij} = \frac{K_I^0}{\sqrt{2\pi r}} f_{ij}(\theta) + \frac{K_{II}^0}{\sqrt{2\pi r}} g_{ij}(\theta), \quad (19)$$

where f_{ij} and g_{ij} are known angular functions and r is the distance to the crack tip. A problem identical to the present with the only exception that the material is homogeneous and given by the Lamé constants μ^0 and λ^0 would be solved by (19). Let a complex stress intensity factor, K , be defined as

$$K = K_I + i K_{II} = \lim_{\substack{x_1 \rightarrow 0 \\ x_2 = 0}} (\sigma_{22} + i \sigma_{12}) \sqrt{2\pi x_1}. \quad (20)$$

Expansions (2a) and (2b) suggest that the stress intensity factor is expanded as follows

$$K = K^0 + \Delta K^1 + O(\Delta)^2, \quad (21a)$$

as $\Delta \rightarrow +0$, where

$$K^1 = K^* + \frac{\mu^1}{\mu^0} K^0 + \frac{\kappa^1}{1 - \kappa^0} \text{Re}(K^0), \quad (21b)$$

with

$$K^0 = \lim_{\substack{x_1 \rightarrow 0 \\ x_2 = 0}} (\sigma_{22}^0 + i\sigma_{12}^0) \sqrt{2\pi x_1} = K_I^0 + iK_{II}^0 \quad \text{and} \quad K^* = \lim_{\substack{x_1 \rightarrow 0 \\ x_2 = 0}} (\sigma_{22}^* + i\sigma_{12}^*) \sqrt{2\pi x_1}.$$

The first order perturbation K^1 is found after inserting (4) and (5) into the last equation, using (11), and recognising that $\sigma_{11}^0 = \sigma_{22}^0$ in the crack plane, $x_2 = 0$, according to (19).

The complex potentials that via (10) provide the solution at large distances from the crack tip are given by

$$\psi = \frac{K^0}{\sqrt{8\pi z}} \quad \text{and} \quad \phi = 0. \quad (22)$$

Further, the stress intensity factor caused by the stress σ_{ij}^* is computed as (Tada et al., 2000)

$$K^* = \frac{i}{\sqrt{32\pi}(\kappa^0 + 1)} \int_G \left[F \frac{z - \bar{z}}{2z\sqrt{z}} + \bar{F} \left(\frac{1}{\sqrt{\bar{z}}} - \frac{\kappa^0}{\sqrt{z}} \right) \right] (d\bar{z}^2 - dz^2). \quad (23)$$

After inserting (17) and (22) into (23) one obtains

$$\begin{aligned} K^* = & \frac{1}{4\pi(\kappa^0 + 1)} \int_G \left\{ K^0 \left[\frac{\kappa^1}{\kappa^0 - 1} \frac{z - \bar{z}}{2r^2} + \frac{1}{\mu^0} \frac{\partial \mu^1}{\partial z} \frac{(z - \bar{z})^2}{2r^2} \right. \right. \\ & + \left. \left(\frac{1}{\mu^0} \frac{\partial \mu^1}{\partial \bar{z}} - \frac{1}{\kappa^0 - 1} \frac{\partial \kappa^1}{\partial \bar{z}} \right) \left(1 - \frac{\bar{z}}{r} \right) \left(1 + \frac{\bar{z}}{r} \right)^2 \right] \\ & + \bar{K}^0 \left[\frac{\kappa^1}{\kappa^0 - 1} \left(1 - \kappa^0 \frac{\bar{z}}{r} \right) \frac{\bar{z}}{r^2} - \frac{1}{\mu^0} \frac{\partial \mu^1}{\partial z} \left(1 - \kappa^0 \frac{\bar{z}}{r} \right) \left(1 - \frac{\bar{z}^2}{r^2} \right) \right. \\ & + \left. \left. \left(\frac{1}{\mu^0} \frac{\partial \mu^1}{\partial \bar{z}} - \frac{1}{\kappa^0 - 1} \frac{\partial \kappa^1}{\partial \bar{z}} \right) \left(2 - 2\kappa^0 + 2\frac{z}{r} - 2\kappa^0 \frac{\bar{z}}{r} \right) \right] \right\} \frac{d\bar{z}^2 - dz^2}{4ir}, \end{aligned} \quad (24)$$

where $r = \sqrt{z\bar{z}}$. The stress intensity factor may be integrated numerically for arbitrary functions $\mu^1 = \mu^1(x_1, x_2)$ and $\lambda^1 = \lambda^1(x_1, x_2)$.

A rectangular inclusion, symmetrically situated with respect to $x_2 = 0$, with a gradient independent of x_2

Here the integration of Equation (24) is performed over a region $G = \{(x_1, x_2) : |x_1 - d| \leq s, |x_2| \leq b\}$ (see Figure 2). After changing to Cartesian coordinates in (24), by using $d\bar{z}^2 - dz^2 = -4idx_1dx_2$ and $r = \sqrt{x_1^2 + x_2^2}$, integrating by parts in the direction of x_1 to remove the derivatives of $\mu^1 = \mu^1(x_1)$ and $\lambda^1 = \lambda^1(x_1)$, and integrating in the direction of x_2 , one obtains

$$\begin{aligned} K^* = & \frac{1}{2\pi(\kappa^0 + 1)} \int_{d-s}^{d+s} \left\{ K^0 \left[\left(\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0 - 1} \right) \left(1 + \frac{2x_1}{r_b} \right) - \frac{\mu^1}{\mu^0} \right] \frac{b^2}{x_1 r_b} \right. \\ & + \bar{K}^0 \left[\left(\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0 - 1} \right) \left(1 - \kappa^0 \frac{r_b}{x_1} \right) - \frac{\mu^1}{\mu^0} \left(\frac{x_1}{r_b} + \kappa^0 \frac{b^2 - x_1^2}{r_b^2} \right) \right] \frac{bdx_1}{r_b^2} \\ & \left. - \frac{K^0}{\kappa^0 + 1} \left(\frac{\mu^1}{\mu^0} + \kappa^1 \right) \right\} \end{aligned} \quad (25)$$

Figure 2. Schematic of a crack approaching a rectangular inclusion.

where $r_b = \sqrt{b^2 + x_1^2}$. As observed in (25) the imaginary terms, apart from K^0 , on the right hand side have cancelled out during the integration. This leads to the expected result that the stress intensity factor for mode I (respectively mode II) vanishes for absent mode I (respectively mode II) load, in case of symmetrically situated inclusion.

An inclusion in the shape of an infinite strip stretching in the direction of x_2

This case follows from (25) for large values of b as compared with d and s , with the assumption that d and s are of the same order of magnitude. The integral should be taken in the sense of Cauchy's principal value. The result reads

$$K^* = \frac{-1}{2\pi(\kappa^0 + 1)} \int_{d-s}^{d+s} \left[K^0 \left(\frac{\kappa^1}{\kappa^0 - 1} \right) - \bar{K}^0 \kappa^0 \left(\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0 - 1} \right) \right] \frac{dx_1}{x_1} - \frac{K^0}{\kappa^0 + 1} \left(\frac{\mu^1}{\mu^0} + \kappa^1 \right). \quad (26)$$

The relations (21) may be used to find the deviation K^1 from the zeroth order stress intensity factor due to the included gradient materials as follows:

$$K^1 = \frac{K_I^0 \kappa^0}{\kappa^0 + 1} \left[\frac{\mu^1}{\mu^0} - \frac{2\kappa^1}{\kappa^0 - 1} - \frac{1}{2} H_I \left(\frac{d}{s} \right) \right] + \frac{K_{II}^0 \kappa^0}{\kappa^0 + 1} \left[\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0} + \frac{1}{2} H_{II} \left(\frac{d}{s} \right) \right]. \quad (27)$$

Here H_I and H_{II} are Hilbert transform of the material gradients μ^1 and κ^1 , defined by

$$H_I \left(\frac{d}{s} \right) = \frac{1}{\pi} \int_{d-s}^{d+s} \left(\frac{\mu^1}{\mu^0} - \frac{\kappa^1}{\kappa^0} \right) \frac{dx_1}{x_1} \quad \text{and} \quad H_{II} \left(\frac{d}{s} \right) = \frac{1}{\pi} \int_{d-s}^{d+s} \left(\frac{\mu^1}{\mu^0} - \frac{\kappa^0 + 1}{\kappa^0 - 1} \frac{\kappa^1}{\kappa^0} \right) \frac{dx_1}{x_1}. \quad (28)$$

Integrals (24) through (26) may be integrated for any pair of known material gradients μ^1 and λ^1 .

5. Results

Results are presented for mode I loading, $K^0 = K_I^0$, and constant Poisson's ratio, i.e. $\kappa^1 = 0$. Plane strain conditions are considered and $\kappa^0 = 3 - 4\nu^0 = 5/3$ is chosen. The results for generalised plane stress differ from the presented ones by a factor of 16/15. The solution for an inclusion with an arbitrary elastic modulus gradient is obtained in two steps. First a solution is found for a half period of a cosine variation of the shear modulus. Because of the linear properties of the solution, a superposition scheme may be employed where the result for an arbitrary gradient is computed via expansion into a Fourier series.

5.1. SOLUTION FOR A HALF-COSINE WAVE

The following perturbation function is selected:

$$\frac{\mu^1(x_1)}{\mu^0} = \begin{cases} Z \cos \left[\frac{\pi(x_1 - d)}{2s} \right], & \text{for } |x_1 - d| \leq s, \\ 0, & \text{otherwise} \end{cases}, \quad (29)$$

where Z takes the values $+1$ or -1 , depending on whether the shear modulus is increasing or decreasing, respectively, inside the inclusion. The solution to the limiting case of an infinite strip is provided by (27). For this case, H_I , defined in (28), after inserting (29) reads as follows:

$$\begin{aligned} H_I(\gamma) &= \frac{Z}{\pi} \int_{\gamma-1}^{\gamma+1} \cos \frac{\pi}{2}(\xi - \gamma) \frac{d\xi}{\xi} = \\ &= \frac{Z}{\pi} \left[\cos \left(\frac{\pi}{2} \gamma \right) \int_{\gamma-1}^{\gamma+1} \cos \left(\frac{\pi}{2} \xi \right) \frac{d\xi}{\xi} - \sin \left(\frac{\pi}{2} \gamma \right) \int_{\gamma-1}^{\gamma+1} \sin \left(\frac{\pi}{2} \xi \right) \frac{d\xi}{\xi} \right], \end{aligned} \quad (30)$$

where $\gamma = d/s$ and $\xi = x_1/s$. For cases with $|\gamma| \leq 1$ the integral is taken in the sense of the Cauchy principal value. Integration of (30) gives after some algebra

$$\begin{aligned} H_I(\gamma) &= \frac{Z}{\pi} \left\{ \cos \left(\frac{\pi}{2} \gamma \right) \left[\text{Ci} \left(\frac{\pi}{2} |\gamma + 1| \right) - \text{Ci} \left(\frac{\pi}{2} |\gamma - 1| \right) \right] \right. \\ &\quad \left. - \sin \left(\frac{\pi}{2} \gamma \right) \left[\text{sgn}(\gamma + 1) \text{Si} \left(\frac{\pi}{2} |\gamma + 1| \right) - \text{sgn}(\gamma - 1) \text{Si} \left(\frac{\pi}{2} |\gamma - 1| \right) \right] \right\}. \end{aligned} \quad (31)$$

The sine integrals, $\text{Si}(t)$, and cosine integrals, $\text{Ci}(t)$, are defined as follows

$$\text{Si}(t) = \int_0^t \sin(\tau) \frac{d\tau}{\tau}, \quad t \geq 0 \quad \text{and} \quad \text{Ci}(t) = - \int_t^\infty \cos(\tau) \frac{d\tau}{\tau}, \quad t > 0. \quad (32)$$

A graphical representation of the solution (27) is shown in Figure 3 for a decreasing shear modulus inside the strip, $Z = -1$. The stress intensity response for an increasing shear modulus is a mirror image of the given graph, with respect to the γ -axes.

Using the gradient given by (29) and integrating (25) numerically, solutions have been obtained for three different values of b/s : $b = s$, $b = 10s$, $b = 100s$. The resulting plots for the stress intensity $K_I^1(\gamma)$ are shown in Figure 4 together with the analytical solution. As seen from the figure, the numerical solutions converge to the analytical, and the curve for $b = 100s$ is practically indistinguishable from the analytical curve.

Finite element (FE) method based simulations provide support to the analytical predictions (Jivkov and Ståhle, 2000). The simulations may also be used to examine the deviation of the

Figure 3. Shear modulus distribution and stress intensity response in a strip region.

Figure 4. Stress intensity variation in rectangular inclusions of various aspect ratio.

results from the analytical due to the adopted perturbation theory. To estimate this deviation a comparison of the minimum, respectively maximum values of the functions $K_{FE}^1(\gamma)$ and $K_1^1(\gamma)$ has been adopted. To underline the fact that the analytical solution (31) is for one half-wave, the resulting stress intensity factor is equipped with a lower index one, i.e. $K_1^1(\gamma)$. The experiments have been run for a set of values of the perturbation coefficient, Δ , namely $\Delta = 0.01, 0.1, 0.5$ and 0.9 for both positive and negative cosine half-wave variation of the elastic modulus, cf. (29). The largest relative difference between the analytical solution and the FE results are presented in Figure 5. From the graphics in Figure 5 the range of validity of the analytical solution is suggested. For example, if an error of 5% is acceptable, the analytical solution is applicable for variations of the modulus of elasticity as large as 40% for a weaker inclusion and 60% for a stiffer inclusion.

5.2. SOLUTION FOR AN ARBITRARY GRADIENT

The solution offered for a problem perturbed by one half-wave, can be extended to resolve the case of any trigonometric elastic modulus distribution by a simple reasoning. Consider, for this purpose, the scheme shown in Figure 6. N half-waves with altering signs describe the

Figure 5. Validity range of the analytical solution.

Figure 6. Illustration of elastic modulus distribution in a form of N half-waves.

variation of the modulus of elasticity inside the perturbation strip of width $2s$. The distances from the central lines of each of the half-waves to the crack tip are denoted by d_n^N , $n = 1, 2, \dots, N$, where the upper index stands for the number of half-waves and the lower index for the number of the half-wave being considered. The half-waves are counted from left to right so that $d_1^N < d_2^N < \dots < d_N^N$ holds. The distance from the central line of the strip to the crack tip is denoted as before by d . It is easy to observe the following relation:

$$d_n^N = d + \frac{N+1-2n}{N}s, \quad n = 1, 2, \dots, N. \quad (33)$$

By introducing the normalised distances $\gamma_n^N = Nd_n^N/s$ and $\gamma = d/s$, Equation (33) becomes:

$$\gamma_n^N = N(\gamma + 1) + 1 - 2n, \quad n = 1, 2, \dots, N. \quad (34)$$

The linearity of the problem allows the solution for the given N half-waves to be found by superimposing the solutions for each of them. Moreover the Hilbert transform is also a linear operator. Consequently the solution is given by:

$$K_N^1(\gamma) = \sum_{n=1}^N (-1)^{n+1} K_1^1(\gamma_n^N). \quad (35)$$

Now a Fourier series may be defined as an expansion of the function $\mu^1(\gamma)$ in a series of sines and cosines such as:

$$\mu^1(\gamma) = \sum_{N=1}^{\infty} c_N \exp\left(i \frac{N\pi}{2} \gamma\right). \quad (36)$$

The expansion coefficients are computed as follows:

$$c_N = \int_{-1}^1 \mu^1(\gamma) \exp\left(i \frac{N\pi}{2} \gamma\right) d\gamma. \quad (37)$$

The solutions (35) can now be combined with (36) and (37) to form a general solution for the case of an infinite strip with an arbitrary modulus of elasticity change. Thus, the deviation of the stress intensity factor may be written in a Fourier series form as proceeds:

$$K^1(\gamma) = \sum_{N=1}^{\infty} c_N K_N^1(\gamma) \exp\left(i \frac{N\pi}{2} \gamma\right). \quad (38)$$

The series is point-wise convergent for any position of the crack tip, except possibly for cases where the crack tip is in a position where the material possess a discontinuous change of the elastic properties.

6. Discussion and conclusions

In the paper, an asymptotic solution for the stress intensity variation when a crack tip is interfering with an arbitrary region of changing elastic properties is derived. A case study for regions of rectangular shapes is considered and specialised to a layer of functionally graded shear modulus. A motivation for not introducing a spatial variation of Poisson's ratio in the case study is that the variation of ν between different materials is rather limited, and in situations such as cracks approaching, running parallel with and on bi-material interfaces the results have been shown to be fairly insensitive to the differences in ν across the interface. If this is the case also for gradient materials, may be a subject for a future study, using the general solution presented in this paper.

An interesting observation, made in Figure 4 for inclusions of weaker material, is the combination of an enhanced local stress intensity factor as the crack tip is approaching the inclusion and a decrease when the crack tip is inside the inclusion. The enhanced stress intensity factor as the crack is approaching, suggests that cracks are attracted by the inclusion and may adopt a path that will lead the crack to the inclusion. Once the crack tip is inside the inclusion the stress intensity factor drops, which may lead to crack arrest. The capability to attract cracks suggests that even a low density of inclusions may have an effect on the fracture toughness.

For an inclusion with stiffer material, an approaching crack will experience a decreasing stress intensity factor. This may lead to crack path deflection away from the inclusion and possibly a small effect on fracture toughness due to the presence of the inclusion. However, the deflection of the crack path can also lead to enhanced fracture toughness via the increase of

the crack surface area. In a material with a low inclusion density the effect may be insignificant whereas the crack will just avoid the few inclusions with little crack deflection as a result.

This leads us to the conclusion that the inclusion density may be of greater importance when the stiffness of the inclusion is larger than that of the surrounding material as opposed to inclusions of weaker material, which may have a stronger effect even at low inclusion densities.

The solution for the special case of an infinite strip (or layer) of spatially changing modulus of elasticity is given in the form of a Fourier series expansion. The selected series converges to any continuous function defined inside the layer. In the case of discontinuous functions, the expansion converges at any fixed position of the crack tip excluding only the points of discontinuity. It is not known, at the present, whether the resulting stress intensity factor will have discontinuities in the form of finite jumps. The solutions obtained for individual terms of a series expansion, provide us with a series point-wise convergent to the result for a discontinuous modulus of elasticity. A truncated series represents the result for a continuous modulus where the discontinuity is replaced with a fast changing wave. A truncated series expansion approximating a discontinuous modulus of elasticity thus supplies a possibility of examining a more realistic case where the jump in modulus occurs over a finite distance. This will give a physical meaning to truncating the series for terms with minimal period. Approximate finite distance over which the jump of modulus occurs should be around one fourth of that minimal period.

The error due to the assumed small changes of the modulus of elasticity is examined only for the leading term of the Fourier series expansion. However direct comparison with numerical results for the leading term can be transformed into results for higher order terms following the method given in Section 5. The numerical procedure for finite changes of the modulus of elasticity (Jivkov and Ståhle, 2000) provides a possibility to estimate the error for any variation of modulus of elasticity. However the result must be treated with judgement whereas the numerical results cannot be superimposed.

7. Acknowledgements

The financial support, coming from the Swedish Centre for Nuclear Technology is highly appreciated.

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