Master’s Degree Project

In

Teaching and Learning in Higher Education

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Teaching with manipulatives for university Mathematics.

Att använda laborativt material i matematikundervisningen på universitetet.

by

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Abstract:

This thesis aims to investigate how teachers are using manipulatives in mathematics at advanced undergraduate level in universities. This will also showcase their usefulness in the classroom. The research is conducted via oral interviews of four teachers in the area. The main conclusions are that manipulatives are useful in that they actively engage students in the classroom, and help them verbalise and see consequences of theory.
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Chapter 1

An early memory of mine was sore fingers from forcing LEGO bricks together to make a stable wall, and eventually constructing that wall with sound structure. We all know that hands on work can help us understand. How far can we take this, can hands on usage of manipulatives aid in the most theoretical of subjects, that of undergraduate mathematics?

1.1 Motivation

The research question of this thesis is to investigate the usefulness and popularity of manipulatives in the class room of advanced undergraduate mathematics students at third level.

In this brief introduction I wish to give the reader the underlying motivations for this thesis and why the author chose this as their research topic.

I am mathematician with an undergraduate in science specialising in theoretical physics coupled with PhD in mathematics. My PhD was in the area of non-commutative algebra, which roughly (in my case) means geometry on surfaces that are not flat (flat meaning internal angles of a triangle sum to π radians). I currently teach in the mathematics curriculum at a school of engineering in Ireland. The majority of my teaching is to engineers that need mathematics modules but who will not study mathematics to degree level. I have taught pure mathematics courses at advanced and masters level and supervised final year and masters level projects. In addition I am a published researcher in the areas of linear algebra and non-commutative geometry. My research interests lay in the fields of geometry and theoretical physics.

Throughout my teaching career I have often noticed lack of engagement being a problem in the classroom. A simple example of this might be a student simply falling asleep during a class or more commonly you will see students asking questions that are not specifically related to the topic at hand. I mentioned this specific case since it will be of special interest in the conclusion section. This is a particularly worrying case of student disengagement because it shows that the student has lost the flow of logic of the current
problem, a key skill in mathematics. I have spoken to colleagues who have taught similar courses in other institutions and they have noticed similar anecdotal evidence the student disengagement. Being the motivated lecturer means that I must endeavor to keep my own students motivated and try and improve the learning outcomes and their learning experience of the course that I am teaching. With this in mind I often strive to try and improve my own teaching methodology. One of the simple ways to improve one's teaching is to make the classroom experience more enjoyable for everyone. This can be accomplished by something as simple as having a nice motivational story having a video to show the class or specific to this research thesis something interactive. After having spoken with colleagues who were teaching in similar areas we often notice that we used to bring in small toys into the classroom to demonstrate concepts or motivations for definitions. A simple example of this that I have used in the past is that of bringing in a small Lego toy and asking students to see if they could predict outcomes based on certain actions. Once you start using one of these toys in the classroom your mind immediately tries to expand upon the topic and see what other type of objects you could use in the classroom. At this point you are naturally led to the consequence of using what are known as manipulatives in the classroom. So this could be something as simple as a Perspex cube showing it to students and asking them to count the number of sides on it. The reason for doing this, might be to verify what is known as the Euler characteristic of the surface, the famous vertices minus edges plus faces always being two formula. This is a very simple example of using a manipulative in the classroom to help students see a particular concept in action and more importantly be able to verify it for themselves.

However less common is the use of manipulatives at higher level in university education. Take for example a course in abstract algebra such as group theory, and now ask some questions: Can manipulatives be used in this type of course?

Do they have any positive effects on the classroom do they help in the learning outcomes Do they help student engagement ? There are so many more. This seems to be an area of lesser study in the literature as the literature is more concerned with either pre university education or early years university education while this thesis is more concerned about advanced undergraduate mathematics third level education and that use of manipulatives in that type of classroom.
Teaching of mathematics in the university sector is mainly based upon the rubric of *definition, theorem, proof*. This is the idea that a concept is introduced then a conjecture formed then proved this is repeated till necessary. While this does accomplish the end goal of learning some mathematics it also has a high dropout rate and tends to quickly leave any less motivated students behind or just ignores them. With this in mind there is and has also been much research in the area of how to engage students better in the subject of mathematics (Eka Sulistyawati et al., 2021).

A common theme of the some of the above references is that of the use of toys in the class room. These toys are often groups as “Science, Technology, Engineering & Mathematics” – or STEM toys. These can range from simple ball and stick models, magnets or interlocking tiles. Here I present a small selection of the types of manipulatives that are common and that I will reference in this thesis.

### 1.2 Sample Glossary of Manipulatives

We start with a classic interlocking, tile style, STEM toy. Here the various tiles can be interlocked in many different ways and tiles usually have different colors. In the picture shown below, is the specific case of interlocking tiles to construct what are known as the five platonic solids. These are the only possible 5 objects whose sides are made of regular polygons that enclose a finite volume. Hopefully it is clear to see that these colored tiles allow students to count vertices edges faces and showcase various characteristics of the solids involved. For example you can quickly see how one might construct the angles at any of the vertices, or if the shapes when laid out in a two-dimensional plane can they tile the plane in such a way that there are no gaps.
Figure 1: The five platonic solids modelled by interlocking tiles. Source: Polydron toys.

Figure 2: Magnetic stick style toy, shown here is a rigid graph type structure than could be used to talk about rigid body forces or minimal distance paths in a graph. Source: Polydron toys.
Figure 2 presents a photograph of an example of magnetic-stick style toys. Here each of the various connections or bars links at a magnetic ball and complex shapes to be constructed in this manner. Variations of these type of STEM toys are commonly seen in chemistry known as ball and stick models. An example of one of these being used in the classroom might be that in the area of graph theory where you are interested in the question of vertices and edges of what is known as a graph network the various colors and linkages in the manipulatives allow one to count vertices and edges hordes terminate certain paths are closed or if loops exist or not.

![Figure 2: A photograph of an example of magnetic-stick style toys.](image)

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The photograph in Figure 3 gives an example of a kinetic manipulative. The image shown here is an example of a spinning top. The usefulness of this in the classroom might be related to the discussion of kinematics, that is the study of motion of rigid bodies. An example of why this might be useful at a higher level of mathematics is that of the phenomena known as precession. This phenomenon allows us to accurately predict orbits of heavenly bodies and other such objects.

![Figure 3: A spinning top, this can be spun at various speeds to demonstrate gyroscopic effects.](image)

*Figure 3: A spinning top, this can be spun at various speeds to demonstrate gyroscopic effects. Source Wikimedia commons*
Figure 4 is an example of a toy known as the towers of Hanoi puzzle. In this puzzle the object is to move the disks from one peg to the other to replicate the tower of ascending discs of increasing radii. The constraint here is that you may only place a disc of smaller radii on top of a disc of larger radii at any one move. The puzzle is now to move one tower to the other side in as small a number of moves as possible. The usefulness of such a toy in the classroom can be seen as follows.

A common question in computer science is that of sorting algorithms and the efficiency of these algorithms. For example a student might be asked to replicate the towers of Hanoi on a computer program. Having the manipulative in their hand allows the student to test to the computer program and to debug it in a physical sense. In addition an instructor could just show the students the problem and then ask them verbally how they might go about solving it hence pseudo coding verbally. In both of these cases the students are actively engaged in the classroom with the problem at hand and have the opportunity to point out errors they may see in the model or offer constructive criticism.
1.3 Scope of the Mathematics in Question

Since this thesis is concerned with a subset of mathematics that is taught at university level. I will now define this in order to assuage any curiosity or concern. We define advanced undergraduate mathematics with the following subject areas:

- Advanced Linear algebra, i.e. matrix algebra, vector spaces, etc

For example, computing the quotient algebra of a linear space, determining eigen values and vectors in a more general context.

- Vector Calculus.

Computations of derivatives in non-orthogonal coordinate settings, beginning of differential geometry.

- Group theory.

Definition, normal sub groups, quotient spaces and Largages theorem.

- Abstract algebra including groups, rings and fields

Rings, fields etc

- Applications of mathematics, for example biological models, financial mathematics etc.

Applications of linear algebra to vision systems, group theory and geometry to mathematical biology.

As an instructor of mathematics to upper level university students, I have encountered some evidence of the use of these manipulatives in these contexts and wished to collect any evidence together in one coherent thesis with the hope of being able to answer the question, what are the advantages of this style of teaching – supporting any conclusions with evidence.

The evidence encountered was in the form of informal conversations with colleagues, anecdotes heard during conferences, social media, and so forth.

1.4 Constraints

This thesis is part of course work, and as such has constraints of time of year and strict deadlines. With these in mind I chose to conduct research by means of interviews with practitioners of this style of teaching. Ideally the conclusions outlined would have been
supported by observations of students in the classroom and interviews with students, coupled with rigours statistical analysis, see Chapter 5.

These concerns and constraints are outlined in further detail in Chapter 5. In addition, this chapter also contains possible avenues for further research in this area.

1.5 Outline

For the aid of the reader, I now outline the basic format of this thesis:

- Chapter 1, Background on the author, roadmap of the thesis, examples of manipulatives and their possible uses, background on the specifics of the nature of this thesis.
- Chapter 2, Literature review of the research in this area to date
- Chapter 3, Methodology of research
- Chapter 4, Finding and themes of my findings from my research.
- Chapter 5, Open questions, and directions for future research.

1.6 A manipulative

As mentioned in the ‘Motivation’ section I aim to study the use of manipulatives in the classrooms of advanced mathematics students. I now explicitly describe what these objects. For the sake of clarity I also define their names here that will be used henceforth in this thesis.

Manipulatives – while this is a generic term for the purpose of this thesis I will use it to define any type of STEM ‘toy’ that is a physical object that can be handled and manipulated. So for example the classic ball and stick magnets or interlocking tiles are common examples. While this defines them, I will also define the converse to avoid any confusion. For example objects such as glass sculptures or 3D art on a poster or computer screen will not count as they either cannot be manipulated or handled by a person.

The advantages of their use is that the manipulatives themselves are generally easy to purchase or even make, and usually affordable. Their downsides are that they are static. That is they cannot easily change colours to show a relationship or similar. Another
downside is that if I have a physical object, I cannot just give you a copy without some layer of cost – unlike a digital asset.

Virtual manipulatives – in recent years the power of modern computing and various sensors have allowed an explosion in the field augmented and virtual reality. There has also been some research in this area such as,(Moyer-Packenham, 2016) This usually takes the form of a student wearing some type of headset with a screen that allows them to either see an object superimposed over their current environment or the object existing in a purely virtual environment. The former being called augmented reality while the latter is virtual reality, Figure 5. The advantages of these are that they can conform to their setting as needed, by changing colours or shapes to help communicate an idea. They are replicable to any degree and easy transmitted over great distances since they are just information.

Figure 5: Example of VR headsets and AR based learning.
Chapter 2

In this chapter I will give a brief overview of the current state of research in the area of mathematics education concerning the use of manipulatives in the class room. Unfortunately the field of mathematics education is heavily biased towards pre university and early university years. This does make sense when you factor in the level of mathematics study that takes place beyond a second year in university. Most graduate courses concentrate on their own needs after the student has a basic grasp of mathematical foundations. With this in mind, the need for research at this level is not as clear as with earlier years, and more importantly researchers well versed in higher level mathematics are less likely to participate in this field or simply are not available. I mention this here as a factor for the lack of citations for specific subjects such as group theory or linear algebra and the over emphasis on subjects more common to early level undergraduates such as basic algebra or calculus. With these factors in mind I now outline a brief review of the current state of the literature for the use of manipulatives in the class room.

2.1 Embodied Cognition and General use of Manipulatives

The theory of embodied cognition is a perspective that emphasizes the role of the body and its interactions with the environment in shaping cognitive processes, such as mathematics. According to this theory, mathematical concepts are not abstract entities that exist independently of human experience, but rather grounded in concrete physical representations, such as gestures, actions that appear naturally when using manipulatives. Embodied cognition challenges the traditional view of mathematics as a purely computational and symbolic domain, and proposes new ways of designing and facilitating mathematical learning activities that engage the body and the senses. (Lakoff & Núñez, 2000) ask the question what mechanisms of the brain allow humans to formulate mathematical ideas.

For example, in the work of Zoltan P. Dienes, as seen in (Bharath Sriraman, 2008), they explore theories on how mathematical structures can be taught from the early grades onwards using multiple embodiments through manipulatives, games, stories and dance are explored. While this is not exactly the same as the research question in this thesis it is a good starting point. For our purposes I will not be exploring the use of stories or dance since they are outside our self-imposed definition of manipulatives. The use of manipulatives has also shown to aid in the understanding of concepts, (Sun et al., 2023). Here
the author is concerned with shape recognition in early learners, aged 5 and 6, with the use of tile based geometric objects. Even in later year learners, the use of manipulatives has also been observed to have positive effects on learning outcomes (Eka Sulistyawati et al., 2021) specific to the STEM areas (of science, technology, engineering and mathematics). The learning outcomes in this case concerned the measure of angles and their relationship to clock time by means of a product called “Magic hour” (A clock face with segmented illuminated circumference). This study is also supported by some statistical evidence by means of a ’t-test’ while not ideal it is suited to the task of the smaller sample size.

Moving on from the above mentions of manipulatives we can see research materials for mathematics closer to something more abstract. Dyah Sinto Rini,(2022) are able to show that the use of so called algebra tiles as an aid to students learning linear algebra. In the study the students were given simple tiles to represent units, then they were tasked to illustrate the concepts of equation solving by means of a weight balance. The research was catalogued by teacher reactions and photographic evidence. As the author mentions, the small sample sized and limited study is not amenable to large conclusions. However they do mention “would affect the class and the impact of something new “. While this may not seem that close to university mathematics it is an important stepping stone.

Linear algebra is the study of vectors, matrices, and linear transformations. It is essential for understanding many areas of mathematics, such as multivariate calculus, differential equations, and probability theory. It also has applications in physics, chemistry, economics, and engineering. Linear algebra teaches students how to solve systems of linear equations and perform matrix operations. We also take time to note the usefulness of manipulatives to aid learners with more specific needs as detailed in (Satsangi et al., 2016). In this study they employed the use of simple weights and balances to help students solve linear algebraic problems in a single variable, with a 90% success rate in solving problems and clear improvement from students baseline results.

2.2 Specific use cases of manipulatives

For the sake of completeness of this thesis and to address any curiosity for the interested reader, we now outline some specific cases for manipulatives in teaching mathematics.
• While geometry might be the most obvious case for manipulatives in the classroom how it is used might not be so obvious. For example, the art of paper folding was in use where the general problem solving skills were the goal (Wares, 2021). The author describes from a teachers perspective the means by which they explored square roots etc. by means of paper folding. The paper while short gives a clean illustration of the mathematics being used along with clear easy to follow steps for a teacher to replication this. Another example was using trigonometric ratios (Wares, 2019). While these studies related to pre university learners the outcomes are still valuable. For example construction of the irrational number system could be easily motivated by the inability of paper folds to show case an angle we know to exist but whose sine and cosine cannot be expressible as a simple rational number. In a similar fashion (“Radical Thoughts on Simplifying Square Roots,” 2013) would also be a good example.

• Non-Euclidean geometry is often an area that is seen as having a high bar to entry for university students, but the use of simple manipulatives can aid in this transition. The well-known Möbius Strip is a prime example of the first example of non-Euclidean geometry for some students. Flórez & Mukherjee et al, (2020) conducted simple experiments on representations of these surfaces as shown below. The experiments detailed, drawing lines along twisted strips of paper, allow us to class this as a manipulative.

Figure 6: The Kein bottle and variation of a Möbius Strip as embedded in 3D space. You can construct your own Möbius Strip by taking a strip of paper giving it one twist and gluing the ends together.
• The use of 3D printers is also a common entry point for those wishing to experiment with the use of manipulatives in their classroom. In (Flórez & Mukherjee, 2020) they explore such uses for the teaching of multivariable calculus in their classroom. Their lesson plan targets three areas. These are Differential properties of multivariable functions, Functions defined on surfaces and Solid regions in R3 and Fubini’s Theorem. This is a specific use case for the more advanced mathematics student at university level and illustrates the use case of manipulatives in the hands of a skilled and motivated instructor. In these cases the overall aim of their study was to give students a gentle physical introduction to non Euclidean geometry and explore its consequences. For example, cutting the band showing Figure 6 along a long centre axis results in two copies of the original band (topologically) interlocked. The results showed that the physical objects led students to discover the new and interesting topologies present, such as trying to count the number of sides of the Mobius band. I think the results are best summarized by the following quote from the study “The students expected to see “just another math presentation” at the beginning, but when we started asking the relevant questions, they seemed to get motivated and offered their conjectures readily.

• Group theory is often a student’s first introduction to abstract algebra at university. Smith, (2016) uses the game of Jenga to communicate the concept of an abelian group, this is one where the product operation is irrespective of order. That is, ab=ba, which is what we usually experience with numbers but is usually not true of compounded operations such as reflections and rotations.

2.3 Conclusion

While the cases given above do clearly support that certain researchers are active in the area of manipulatives in the classroom, it is also the minority of researchers. In completing this section the author struggled to find the above examples for two main reasons.
First there is simply a lack of research in this area as it relates to the context of university level mathematics education. Second was the overwhelming majority of articles that were specific to either special needs, preschool or preteen students. When doing any internet database search these seem to flood results either due to inappropriate titles, abstracts or searching terms. Thus, there is clearly a need for more research in this area.

With the above in mind this study will add to the current status of literature in the use of manipulatives in mathematics at the university level. This thesis is intended to have two main purposes. First to show case how teachers are currently using manipulatives in their classrooms at advanced undergraduate level, this will have a corollary of highlighting how teachers are using manipulatives for advanced undergraduate mathematics. Secondly I hope to add to the existing catalogue of manipulative usage in the literature and act as a pathfinder for others wishing to adopt this teaching method.
Chapter 3

Methodology

There are many questions that we could ask for this thesis concerning itself with the broad areas of manipulatives in the context of embodied cognition. The focus of this thesis will be, the utility of manipulatives in the classrooms of teachers of advanced university mathematics as evidenced by those teachers. Given the constraints outlined in the previous chapters the methodology will be interviews with teachers of mathematics using manipulatives in their class rooms. The interviewees were all currently employed in third level teaching of mathematics in Ireland, and all had teaching experience outside of Ireland also. These interviews were conducted between February and May of 2023, the start of April being a common end of term time in Ireland.

A full treatment on qualitative research can be found in “Qualitative data a methods source book” by Miles et al (2014). Given the time constraints and small number of interviews here I at first conducted the interviews using my own intuition rather than studying formal texts and applying their knowledge. This was not a wise choice since I was constantly in the situation of self-correcting and augmenting to account for any failings such as missing follow up questions or simply allowing the interviewee to talk instead of interjecting. I would have preferred to study the known methods before conducting interviews. These range from some simple planning ahead to anticipate follow up questions to implementing a flow chart as seen in (Wengraf, 2001) to help guide myself to target the research objectives.

The central question I sought to answer here is:

What is the utility of manipulatives in advanced university mathematics teaching?

Of course, in answering this basic question I will also develop ideas and suggested answers to other questions such as:

- What is the current state of the teachers awareness of manipulatives in the field of advanced undergraduate mathematics?
- As observed by the teacher, how is student engagement changed by their use?
• *What are our metrics for examinations and are they sufficient?*

While the above central question is sensible it still allows for much wiggle room specifically we are aiming to address the following points:

• Constraints and affordances of this method of teaching as experienced by the interviewed teachers.
• Do the interviewed teachers notice any effects in the context of embodied cognition
• Any effects of this style of teaching, unknown to the author in the least or the teaching community at the most.

This chapter is concerned with the data collection method.

### 3.1 Personal experience

This was my first attempt at conducting qualitative research in the form of oral interviews. On the face of it this seems like a straightforward task however it proved to be more complex than I initially expected. For example, when interviewing a peer or colleague there is a certain degree of interplay between professional relationships and personal relationship. When conducting the interview, you both need to be probing in terms of your questions but also respectful in terms of the other person's time and allowing the other person to voice their opinions. Moreover, it is important to design questions in such a way that allow the interviewee to expand upon their answers without feeling like they are being interrogated. It was this latter aspect that the author found the most challenging as it was a new dynamic of social interaction that they were not used to. It should be noted that the interviews were conducted over the course of one month with small tweaks to question style and interview technique that were learned along the way. As mentioned above a guiding text such as seen in Creswell&Poth would have paid dividends to myself.

### 3.2 Data collection
My data will be comprised of answers to interview questions with lecturers in the university sector. Each interview lasted no longer than 1 hr with a median time of 30 – 40 min. In total four interviews were conducted.

I did not consult any standard text before proceeding to conduct interviews, my only guidance was from my supervisor. In retrospect I would have liked to have become familiar with some reference materials. The flow charts seen in (Wengraf, 2001) make for an easy and user friendly method for keeping interviews on track.

The framework for each interview was a video call over Teams or Zoom. During the call, I simply asked the questions and let the person talk and interjected for any points of clarification or follow up questions that may have arisen. I see the relatively short questions as a jumping off point in the hope that the interviewee will expand upon their own personal experiences in trying to answer the question. I see this approach having some pros and cons:

- Trying to ask an open ended question and waiting for their response does allow for the possible serendipity of discovering something that may have never occurred to myself.

- Allowing the respondent to talk freely in my experience usually leads them to start talking and then while talking start to draw connections and reasons for why one thing was done. Personally I think this is just forcing the person to verbalize what they already know or do on a subconscious level. Direct questions do not allow for this.

- I have found that trying to craft questions is difficult, on one hand I find it hard to know how open ended is a question without it being so open ended that the reply is of little use, while on the other end to direct and the reply is short and pithy.

- There is a social aspect to the interview that I was not prepared for, you almost have to make them feel comfortable to open up - a skill I need to learn or one that feels odd to apply in a professional context.

3.3 Interview Questions
A good primer for interviewing is (Brinkmann, 2022) and (Orphan, 2018). For example very simple observation such as it is better to memorise your questions compared to riffling over pages mid interview. Another specific recommendation is to have a two column single page of paper, with questions on the left and research themes on the right. Below are my interview questions, with room allowed to expand, but these are the minimum that were asked, with italics to explain rational.

1. Tell me about your use of manipulatives in the classroom. – *An opening to help them feel at ease and naturally open up about the topic without feeling the pressure of right or wrong answers.*

2. Why did you start to use them? – *Need a timeframe since this might be needed to interpret any effects of this in the classroom.*

3. Did you conduct any research formal or otherwise into the use of manipulatives before starting to using them? – *I wanted to know was this just based on feelings or empirical research, and if so how versed were they in this field of education.*

4. Can you give an example of something that you find easier with manipulatives and something that you do not find easier with them? – *A question to get concrete examples that I can cite for evidence, and similarly to find areas that they did not enjoy, possibly to compare and contrast with others.*

5. Outside of course grades do you notice any difference pre and post usage of manipulatives? – *I knew numeric grades would be less telling hence the need to ask this.*

6. Were numerical/letter grades effected? – *Needed for evidence*

7. Do you have examples of how a student’s perceptions of a problem have changed? - *Needed for evidence as examples to compare and contrast*

8. In the future how do you see yourself using these in the classroom? – *To see if this was positive impact in the classroom.*

9. Would you like to add anything to be documented regarded the use of manipulatives? – *An open ended question to pick up any other data points.*

3.4 Augmentation
During the process of conducting these interviews there was a certain degree of self-reflection and self-learning in the process. For example during the first interview I had a common issue where I was asking multiple questions inside one question, e.g.

*When did you first start using manipulatives and why?*

In fact question also raised another issue. The point of concern was the directness of the question – and forcing the interviewee to give a direct answer such as “last year because I was told to” this of course lacks any context and omits any serendipitous data.

For these reasons I embarked upon my own learning journey in the compiling of this thesis. The main points learned were the following:

- To be clear that this a is a professional conversation with an end goal.
- To know how to politely and respectfully reach said end goal in the time allotted while still allowing the interviewee to express themselves.
- To understand how to phrase a question that both seeks an answer to a specific question but also permits the interviewee to offer up unknown details that may wish to be expanded upon.

For the interviews that were conducted here, no two interviews were identical but all tried to focus in on the key question I wished to answer.

### 3.5 The Interviewees

Given the nature of how the research was conducted this is a qualitative based study and findings here lack any rigorous statistical backing. Hence for this reason numbers, count data and any numeric statistics are omitted.

For confidentiality reasons I will pseudonymize our interviewees as “Sarah” “Michael” “John” and “Rachel” – as mentioned before these are all currently teaching in the university sector of Ireland in a mathematics program. As preliminary I give a short biography for each person that is relevant here:

- Sarah – mid career; usually teaches basic calculus, computing and third year group theory, has published in the area of mathematics education infrequently. Sarah did a PhD in the area of analysis and current has one masters student under supervision.
• Michael – early career; usually teaches statistics, differential equation solving and calculus. Michael obtained a doctorate in the area of applied material science. They are exploring funding options to take on research students.

• John – early career; teaches statistics to non-mathematics students and group theory. John has both a doctoral degree in statistics and industry experience in the pharmaceutical sector.

• Rachel – mid-career; teaches group theory and computing to third and final year students. Rachel started in the area of abstract algebra with her PhD but now mainly works in the area of computations. She has supervised a PhD student in the past.

We define early career as being less than 15 years since PhD, mid-career as being between 15 and 25 since PhD. Sarah and Michael were working in the same university, and John and Rachel together in another university. I note that Rachel has worked in another university system in the Americas in the past 5 years. The author wishes to thank these people for them offering their time to help complete this study.

3.6 Analyzing of the Data

Each interview was conducted via Zoom or Microsoft Teams, and recorded in a video file. The interviews lasted approximately 30-40 min each. After the first interview the file was transcribed and analyzed – at this point the author identified any issues or follow ups that might have been missed and augmented the following interviews, this was an iterative process that led to the final collection of transcriptions.

At this point under guidance from my supervisor and after consultation with (Braun & Clarke, 2022) a thematic analysis was conducted. To accomplish this, the text from each interview was broken in section corresponding to identified themes. Then quotations and text were aligned to themes and finally the information was collated in a spreadsheet. Given the small sample size there was no automated or database software used and everything was manually sorted. The details of themes are discussed in the next chapter.
Chapter 4

I now present my findings, this will be in the form of identified themes that were observed along with supporting evidence to answer the question of how the teachers I interviewed were using manipulatives in the classroom.

4.1 Usefulness of manipulatives in the classroom

Every interviewee explained that they noticed more active engagement in their classrooms that could be attributed to the use of a manipulative. This was clear to some since the early stage of their course by its nature was definition heavy, but once applications were introduced with the use of manipulatives they noticed students more willing to ask questions. For example Sarah mentions that "I noticed now that students are a lot more willing to ask and a lot more willing to challenge answers." Similar type replies were mentioned by all. The main reason for this was again signposted by Sarah when she mentions "they can actually point to and see the object that might have a ramification from the theory." John mentioned the following: "they seem to ask more appropriate questions."

In all of the above the usefulness of manipulatives in the classroom was used after the introduction of a topic or definition such as symmetries, or similar. Rachel cited the use of a particular manipulative known as the Towers of Hanoi (figure 4). She used this to help students better grasp the need for sorting algorithms in a computer science class. Here the manipulative was shown to the class and then they were tasked to solve the puzzle of moving disks to a tower according to the rule that the disk can only be placed in ascending order by radii. It was observed that this led more natural conclusions about what definitions would and more importantly would not be appropriate. Once this lesson was concluded the formal definitions and examples were introduced. After this Rachel, similar to John noticed that questions were more focused and appropriate and grounded to the theory at hand.

While the above points to the usefulness of allowing students to see definitions and have a better understanding, the interviews also highlighted how the manipulatives
aided in challenge based learning. For most this involved either splitting their class into groups and tasking them to present a small example of the consequence of a definition or theorem. For this the stakeholders in the process were the other students that they were presenting to. It was observed that in this situation that the stakeholders (the students in the audience) were more confident or felt empowered to ask questions and challenge assumptions. Sarah mentioned this with “a lot more willing to challenge answers given that they can point to an object.” This highlights a significant improvement in the teaching dynamic where students feel empowered to challenge and ask critical questions. This can be interpretated from the data with quotes such “…students seem to ask more appropriate questions.” And “…they became a little more creative in their problem solving.”. This was cited as making the teaching experience more enjoyable for both students and instructors.

In the context of usefulness the final subtheme is the concept of ‘fun’ in the classroom as mentioned by both Sarah and Michael. Michael observed that class feedback forms at the end of term had a common theme that the students enjoyed their classroom time and the exercises that were based upon the use of manipulatives in the classroom. All but one interviewee remarked that they found the use of manipulatives helped motivate them in the preparation of course materials since otherwise it was very similar and frankly repetition of what they done the previous year. John did give a counter point here that use of manipulatives can be a time consuming task and as he put it ” sometimes more hassle than it was worth.”

This section ends with a short summary of my findings that help answer the main question of this thesis:

- Manipulatives were seen to be helpful to the teachers I interviewed in the classroom in terms of student motivation, enjoyment and engagement.
- The manipulatives helped to motivate definitions and concepts and aided in challenged based learning since students felt empowered to challenge answers or assumptions.
- Lecturer enjoyment was cited by most as it was different from what the traditional classroom environment was.

4.2 Lack of awareness in the field
As detailed in their biographies, none of the interviewees were based or educated in the field of mathematics education or had education backgrounds. They were all pure mathematics undergrads who pursued mathematics to graduate level and now lecture it. This seems to have caused a certain fear or voluntary blindness to the educational sector. Every interviewee when asked what background research they conducted, made a point to say that they had not consulted any of the educational literature nor had done any searching for it. The most background research that was conducted was that of asking someone with knowledge in educational sciences their opinions and this was only motivated since they knew each other personally.

4.3 Engagement in the multimedia age

As you can see from above none of the interviewees were late stage career teachers. This seems to play a role. The evidence for this was that every person responded to the “what motivated you to start this” question with a negative experience of their own “definition theorem proof” style education. In all cases they genuinely seemed concerned that their students would suffer the same experience and wanted to do everything within reason to give them a reason to find the class as enjoyable as one could possibly expect.

It was only the early stage career person that cited conversations with colleagues or journals as a possible motivating factor but in these cases it was a minor factor. Interesting of the early stage researchers they all cited the use and consumption of ”YouTube” content. This has two sides to it. First they are all well aware that students use and sometimes abuse the internet as a resource, so for this reason they try to be aware of any mathematics content trending online. For example, a number of years ago a common maths trick was to show that ”1+2+3+.....=-1/12” (a simple abuse of a divergent series) The second side comes from the interviewees own personal consumption of YouTube content. In one case, I was shown the YouTube history of my interviewee and was greeted with a wall of trending mathematics videos. Some of these were the channels ”Veristastium”, ”Stanupmaths” and ”ThreeBrownOneBlue”. The latter of these was cite by others too. In my own investigation of this youtube channel they make very impressive use of computer animations and visuals. While it is outside the scope of this study
to describe how youtube and other media influence mathematics instruction at the university level, my respondents appeared were all aware of it. Ideally I would have had a larger collection of interviewees with a wide collection of backgrounds upon which to draw a conclusion, but in it’s absence I will just make the remark that they were aware of the influence of modern media.

![Figure 7: A frame from the YouTube channel, three brown one blue video on Fourier series. Note the use of colour, objects and multiple viewpoints. Source “3Brown1Blue”](image)

So in summary my research leads to conclude that the following are the main factors responsible for the use of manipulatives:

- Wishing to retain student interest, and not wanting students to experience the old style of teaching
- The ubiquity of well-produced modern media showcasing mathematics from many different viewpoints, commonly seen on YouTube.
- Peers or journal articles seem to play little to no role, but it should be remembered that consumption of mathematics education journals would not be common to any of the interviewees.

### 4.4 Assessments

Measuring the impact or usefulness of manipulatives in the classroom by its nature would lead one to ask above final grades in courses. At this point I pause to remark that this study was conducted in the Irish education sector and in Ireland, a final end of term,
closed book exam is the usual method of assessment. It is uncommon to see orals or other non-tradition forms of examination take place.

With this in mind every interviewee replied that the final grades in their course that used manipulatives in the class room were not effected in any way that was statistically measurable. Two possible reasons for this were raised by most of the interviewees:

- Covid-19 was a large impacting factor in the past two years, and thus much of the recent knowledge is tainted by this. Covid restrictions in Ireland did not all for mass in hall exams and most courses reverted to 100% continuous assessments online, which made high resolution data of results hard to find.

- The second and more interesting and in the authors opinion far more important issues was that assessments in mathematics are largely concerned with quantitative questions, I now expand on this issue.

Most of the responses mentioned that student understanding and student questions improved after seeing how to use manipulatives in the classroom. In early level university mathematics a student is generally asked to solve a problem or compute and number, while this is an important skill it is not the core skill needed in mathematics. The ideal mathematician is a logical problem solver that can break a given problem down, see connections between problems and call on various facets of their learning to pose a possible solution. Clearly this is not something that can measured as a skill with ease during a 2-3 hour final end of term exam. Both Sarah and Michael remarked that they felt the students had gained a better understanding but yet did not see it in numeric grades – however Michael did give the following remark ”it gave them more insight into how to go about problem solving”. So here we have some evidence that use of manipulatives did indeed help the students in their understanding yet the system of assessments did not allow them to show this. This suggests that the way students are assessed is not an accurate picture of their knowledge and skills, and therefore an issue which needs to be addressed. Ideally an assessment rubric would be one that showcases a students’ ability to deploy the tools of mathematics for computation and showcases their understanding of the core theory. This could be accomplished by a viva style exam setting. However in classes whose size is larger than 10 students, time and logistical constraints might not allow for this with any ease.
4.5 Examples of things better or worse with manipulatives

We next arrived at the questions of what type of content was suited or not suited to the use of manipulatives in the classroom. I start with the former. For the interviewers who were teaching any type of abstract algebra, namely group theory or fields this was a simple question to answer. I said this since most books that introduce group theory start with a simple geometric shape and ask you to count the number of symmetries.

![Figure 8: A simple example of the symmetries of a triangle](image)

After this the student is then after to repeat for a three dimensional shape such as a tetrahedron, such as the one shown in Figure 8. At this stage it hopefully very clear why a manipulative is a useful object the in the classroom.

Two of the interviewees that worked together and had both used these in the group theory classes remarked upon an expanded use of the manipulatives. They, after discussing with each other before this research used manipulatives to help explain to students the consequences of Lagrange’s theorem and how the student could verify it for themselves.

Without greeting too deep into the mathematics, the theorem states that one number should be divisible by another, where each number counts a certain number of symmetries. Clearly the student can be tasked to count the number on a manipulative can verify the count for themselves by seeing if division is possible (division meaning zero remainder).

For those who were teaching in the computer science areas, the towers of Hanoi seen in Figure 4 was also used. This is a common exercise in a computer science class to help
learning sorting algorithms. The lectures were able to physically debug certain coding issues by illustration to the students.

The one type of teaching that is common and was always cited as not being amenable to manipulatives was that of calculus. No interviewee was able to think of a way to take a physical object and somehow apply calculus to it. Calculus generally involves sequences and limits, this in turn would mean an object that is divisible to smaller and smaller replicable of itself. This makes the author think of a Russian Doll, but while this is a nice image it would not hold up as useful outside of an image in a classroom. At this point the author wishes to mention the instances of manipulatives for the teaching of linear algebra, and graph theory. In for the former, simple rubber objects were used to illustrate stress and strain directions that correspond to Eigen vectors, while in the latter the ball and stick style magnets were used to illustrate the various path connected components.

4.6 Conclusions

It was very clear that manipulatives are very useful in university mathematics, according to the perspectives of the four interviewees. The common theme cited as to why they are useful was that of more active engagement by the student. Quotes such as “…they became more creative in their problem solving” – Michael, “…alot more willing to engage…see a real world example” – Rachel, show us this. Furthermore students gain confidence in asking questions when they are able to point to an object. The idea of making the classroom a more ’fun’ environment was also cited but trying to define ’fun’ seems difficult if not oxymoronical.

In Section 4.2 I note the lack of awareness of pedagogical research in this area. After reading (Berliner, 2002) there is a disconnect between the so called hard sciences such as mathematics and the so called softer sciences such as education. The part of educational literature that concerns itself with mathematics is already a subset, when restricted to university level and again to advanced university level topics such as abstract algebra and geometry there is a very small volume of research available and probably none in the exact area of interest. I also note that non of the interviewees have taken any formal pedagogical courses, of the four the only contact researchers in the area of education was from professional friendships. Moreover there is a professional problem too. From my personal experience if you are an educational researcher who works in the
field of mathematics education there is a good chance that your formal mathematics education finished around first or second year of university and you went on to pursue educational research. Of course this is just my experience. Simply put people do not have the bandwidth to study both topics to research level. With this in mind, it is understandable that most mathematics education research concerns itself with early undergraduate materials such as calculus and simple linear systems and there is a lack of materials concerning advanced topics. The author feels there is space for professional mathematicians to contribute to this area. A possible barrier to entry is the very different research style. For example this volume of text would frighten most mathematical researchers, the author included.

Another topic that seems to have been a factor in the lack of background researching was the extra effort in terms of accessing this information. Most interviewees who tried to look for some research just did a simple Google search or Google scholar search in the hopes it would link them to something useful. In the mathematics community the two single most import research locations are the Arxiv and Mathscinet – the former being community driven and the latter being an index of peer reviewed papers. To put it simply the extra step of going to your universities library search engine is just an extra step that adds extra effort on the user side when compared to what a mathematician knows and loves. From the personal experience of the author most university library systems are lacking or simply confusing in their implementation perhaps because they are trying to serve so many difference research communities.
Chapter 5

As mentioned in Chapter 1 and Chapter 2, this thesis is not intended to be a definitive text on the subject of manipulatives in the teaching of mathematics at university level. Instead it should be viewed as an initial foray into laying a path into a more comprehensive research project.

My thesis asked the question, what are teachers experiences of using manipulatives with advanced undergraduate mathematics students. Based on the evidence I have gather I can answer that they do. Moreover I have observed that they aid in student participation and engagement as a whole. This is evidenced by all interviewees citing that students were more engaged in the classroom, and asking more informed questions on the topics. The piece of evidence that most supports this was from Michael. He mentioned that after manipulatives being used to introduce a concept students were able to approach problems without the prompt of ‘using the Sylow Theorem solve the following’. Instead they understood the natural use of the theorem and how to apply it. My work supports the findings of Dyah Sinto Rini,(2022), namely that manipulatives can be used to teach algebraic concepts, and shows that this is not confined to pre or early year university education. Similarly by the evidence collected I have highlighted novel uses of manipulatives such as teaching Lagranges theorem in group theory. This adds to the collection of methods such as those cited in Wares 2019 and 2020. In other countries the concepts of written and oral exams are used together to measure a student’s ability.

No interviewee reported that manipulatives made anything worse, but this was largely due to them not using them unless they had a clear plan and were confident of their role in the classroom, which only reports that the interviewees were competent in their job in the least.

Lastly I noticed it is common for mathematics departments to be seen as adjacent to the other sciences such as chemistry and physics since they both deal with numbers and trying to prove or disprove statements. This assumed closeness usually has the effect of
distancing mathematics departments for others, such educational departments. Over time a group think mentality occurs and professional distancing occurs and hence the lack for cross pollination of research opportunities. Evidence for such a divide can be found in (WATKINS & TEHRANI, 2020)

With this in mind, the remainder of this section will comment on the possible future avenues of research in this area. These will be broken into two main areas, specifically qualitative and quantitative studies.

First as mentioned in the chapter relating to data collection this data was collected as the completion of one part of a course thus necessitating certain time constraints. I would have liked more time to conduct more substantial interviews perhaps maybe with up to 20 instructors. With this level of interaction a more comprehensive coding of the interview data would have been required thus necessitating better software used for the parsing of this data. Again this would be work for the author to learn how to use this software and properly code parsed such large datasets. With access to a larger number of instructors a better feeling for questions and follow up questions would have been expected. Ideally the author would have liked to have interviewed a specific number of instructors who teach in specific areas of mathematics such as algebra analysis statistics and applied mathematics. In fact the area of applied mathematics is completely ignored in this thesis save one picture of the spinning top example in chapter one. I now outline some specific goals for future study that are more qualitative.

- A larger group of interviewees, with software training on parsing their responses.
- A study of the students themselves and how they found their courses.
- A study by means of structured interviews with students that asks them to problem solve before and after having been exposed to manipulatives in a learning environment.
- A study of assessment with emphasis on outcomes from standard numeric grades and teachers evaluation.

In addition to the above qualitative avenues, I would also like to explore quantitative analysis, specifically both teachers experience in this area and also student engagement – the latter having been touched upon here but a much wider and in depth topic. By its nature this will involve sample sizes ideally above 30 or less than 30 once care is taken
with t-tests and the like. Clearly this requires more logistical work and planning that would intersect with term time better. With the above in mind I would wish to carry out the following:

- A survey of instructors by means numeric data on their opinions on the use of manipulatives in the class room, so for example questions such as “Do you find them useful?” with 1 being the lowest and 5 being the highest.
- A similar survey of students, perhaps conducted at the start and the end of term.
- A survey of grades achieved by students from classical written exams and oral exam – with exposure to manipulatives.
- A survey to compare mathematics teachers across levels from pre university to final year undergraduate courses.

While the above is clearly a very long list of objects there is hopefully some subset that is achievable.

Conclusion

To finish this thesis, I think I have answered the question that manipulatives are useful in the classroom with a positive based on my study. However, there is clearly much more research needed in the area.

The quote “students seem to ask more appropriate questions”, resonated with me. A common complaint among third level teachers is that students sometimes obsess over the wrong thing and gloss over the important thing. For example a common situation, from my own experiences, is the following. When showing students how to solve a certain equation, the text of the solution might span three or four pages. While the method of the solution is identical to the example that took half a page, the only difference is that the latter involved awkward. From a personal viewpoint I view this as a major positive from the usage of manipulatives in the classroom.
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